DESIGN OF RESISTIVELY MISMATCHED LINEAR ACTIVE TWO-PORT

A Thesis Submitted
In Partial Fulfilment of the Requirements
for the Degree of

MASTER OF TECHNOLOGY

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BY

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VPNamjoshi
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LIST OF SYMBOLS, ABBREVIATIONS, SUFFIXES

Symbols:

The page number refers to the page on which the symbol appears first.

a	$=$	$\dfrac{\sin\theta}{n}$, p. 34
b	$=$	$-\left(1 + \dfrac{\cos\theta}{n}\right)$, p. 34
c	$=$	$-b = \left(1 + \dfrac{\cos\theta}{n}\right)$, p. 54
C_1 to C_4		Asymptotic error constants, p. 63
δ		Invariant alignability factor, p. 25
δ_i		Invariant inherent alignability factor, p. 26
$\triangle$		$\dfrac{c^3}{27} + \dfrac{a^2}{4}$, p. 37
Denom.		Denominator of F, p. 74
ϵ		$\dfrac{n - \Pi}{\Pi}$, p. 78
$\mathcal{E}_x$		Error in x defined as: (calculated value of x − exact value of x)/exact value of x.
F		Normalizing Factor, p. 36
$f_1(\lambda)$ to $f_4(\lambda)$		Iteration Functions, p. 61
g_1		Loop gain, pp. 3, 5
g_{1c}		Loop gain with conjugate matched terminations, p. 31
g_{1r}		Loop gain constrained to be real, p. 30

g_p	Operating (transducer) power gain, p. 33
$g_{max\ n}$	Maximum value of operating power gain for a given performance factor, n, p.36
η	Invariant factor, p. 24
η_i	Invariant inherent factor, p. 24
θ	$\arg(p_{12}p_{21}) = \tan^{-1}\left(\frac{N}{M}\right)$, p. 19
k	Stern's stability factor, p. 22
k_i	Stern's inherent stability factor, p. 23
M	$\mathrm{Re}\left[p_{12}\ p_{21}\right]$, p. 19
N	$\mathrm{Im}\left[p_{12}\ p_{21}\right]$, p. 19
n	Performance factor, p. 24
n_i	Inherent performance factor, p. 24
L	$\lvert p_{12}\ p_{21} \rvert$, p. 19
λ_o	Optimum root of the cubic equation, p. 34
λ_{oe}	Exact value of λ_o, p. 56
λ_1	$\frac{\sigma_1}{\rho_1}$ (Between p. 28 and 36 only)
λ_2	$\frac{\sigma_2}{\rho_2}$ (Between p. 28 and 36 only)
$\lambda_1, \lambda_2, \lambda_3$	Three roots of the cubic equation (3.15) or (4.1), p. 37
$\lambda_{o1}, \lambda_{o2}, \lambda_{o3}$	Three successive approximations to λ_o, p.68
${}^1\lambda_{o1}$	Approximation to λ_o obtained after using δ^2 process, p. 70

$\begin{bmatrix} p_{12} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$	General two-port matrix where p = h, y, z or g, p. 5
p_S	Source immittance, p. 5
p_L	Load immittance, p. 5
p_1	$p_{11} + p_S$, Total self-immittance at input port, p. 5
p_2	$p_{22} + p_L$, Total self-immittance at output port, p. 5
p_{in}	Input immittance, p. 17
p_{out}	Output immittance, p. 17
$\prod$	$\dfrac{1 + \cos\theta}{2}$, p. 24
ρ	$\mathrm{Re}\left[p\right]$
σ	$\mathrm{Im}\left[p\right]$
S	Invariant stability factor, p. 25
S_i	Invariant inherent stability factor, p. 25
T	$\dfrac{a^2}{c^3} = \dfrac{n \sin^2\theta}{(n+\cos\theta)^3}$, p. 54

Abbreviations:

a.c.c	Asymptotic error constant, p. 62
LHP	Left half of complex plane, p. 13
SC	Short-circuit, p. 14
OC	Open-circuit, p. 14
Nr	Numerator, p. 33
Dr	Denominator, p. 33

Suffixes:

e	exact value
o	optimum value
S	Source termination
L	Load termination

SYNOPSIS

DESIGN OF RESISTIVELY MISMATCHED
LINEAR ACTIVE TWO-PORT
A Thesis Submitted
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In high frequency tuned amplifiers, a serious problem is that of oscillation while tuning the amplifier. Two methods of eliminating these oscillations are i) Unilateralization and ii) Mismatch. In the latter method, the resistive parts of source and load terminations have magnitudes different from those needed for conjugate matching. In the design of resistively mismatched amplifiers, the operating (transducer) power gain is maximized for a given value of performance factor, n (a stability-based factor) the value of n being chosen so as to ensure absolute stability of the two-port. While maximizing the operating power gain for a given n, a cubic equation results viz.

$$\lambda^3 + \left(1 + \frac{\cos\theta}{n}\right)\lambda - \frac{\sin\theta}{n} = 0$$

where $\theta = \arg(p_{12}\, p_{21})$, $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ being the general two-port matrix. The appropriate root λ_o, of this equation must be found before the reactive parts of the source and load terminations can be calculated.

In the present work, simple approximate expressions for calculating λ_o and the maximum optimum power gain $g_{max\ n}$

are given. It is shown that λ_o has the same sign as $\sin\theta$ and that

$$\text{for } n \geqslant 5, \quad \lambda_o \simeq \frac{\sin\theta}{n+\cos\theta} \quad \text{within } 5\%$$

$$\text{for } n \leqslant 0.1 \quad \lambda_o \simeq \sqrt{\frac{1-n}{n}} \quad \text{within } 3\%$$

For $0.1 < n < 5$ an iteration method is shown to yield the value of λ_o within 1.5% in 4 computation. The maximum ~~operat-~~ operating power gain is shown to be

$$\text{for } n \geqslant 5 \quad g_{max\ n} = 4\,\frac{n}{n^2-1}\cdot\left(\frac{n+\cos\theta}{n-\cos\theta}\right)\left|\frac{p_{21}}{p_{12}}\right|\left\{1-\sqrt{\frac{n_i}{n}}\right\}^2$$

within 0.15% ($\simeq$ 0.01 dB)

$$\text{for } n \leqslant 1.3\,\Pi \quad g_{max\ n} = \frac{4}{\epsilon^2}\left[1+\frac{3-\Pi}{4}\,\epsilon\right]\left|\frac{p_{21}}{p_{12}}\right|\left\{1-\sqrt{\frac{n_i}{n}}\right\}^2$$

within 3% ($\simeq$ 0.13 dB)

$$\text{where} \quad n = \Pi\,(1+\epsilon) = \left(\frac{1+\cos\theta}{2}\right)(1+\epsilon)$$

For $n < 5$ but $> 1.3\,\Pi$ the value of λ_o must be directly substituted to give

$$g_{max\ n} = 4\,\frac{n}{\left[n(1-\lambda_o^2)-\cos\theta\right]^2+\left[2n\lambda_o-\sin\theta\right]^2} \times \left|\frac{p_{21}}{p_{12}}\right|\left\{1-\sqrt{\frac{n_i}{n}}\right\}^2 \quad \text{within } 3\%\ (\simeq 0.13\text{dB})$$

CHAPTER 1

INTRODUCTION

1.1 Loop Gain:

A serious problem in RF and IF amplifiers is that, when one varies the source and load reactances in the process of tuning, the amplifier invariably oscillates. This is due to the internal feedback in the device (tube or transistor). Sufficient energy is fed back at the input terminal to cause oscillations. Feedback also makes the input immittance a function of the load immittance and the output immittance a function of the source immittance. This makes the alignment of multistage IF amplifier difficult. The internal feedback of the device may be visualized by a black box approach.

Consider an amplifier, utilizing a device (transistor) in y-environment, that is, with shunt source and load terminations. This is represented by Fig. 1.1. Then the amplifier, henceforth referred to as a two-port network, may be represented by its y-equivalent circuit of Fig. 1.2. Then

$$\begin{aligned} i_1 &= y_{11} v_1 + y_{12} v_2 \\ i_2 &= y_{21} v_1 + y_{22} v_2 \end{aligned} \qquad \ldots \quad (1.1)$$

But

$$i_S = y_S v_1 + i_1 \quad \text{and} \quad i_2 = -y_L v_2 \qquad \ldots \quad (1.2)$$

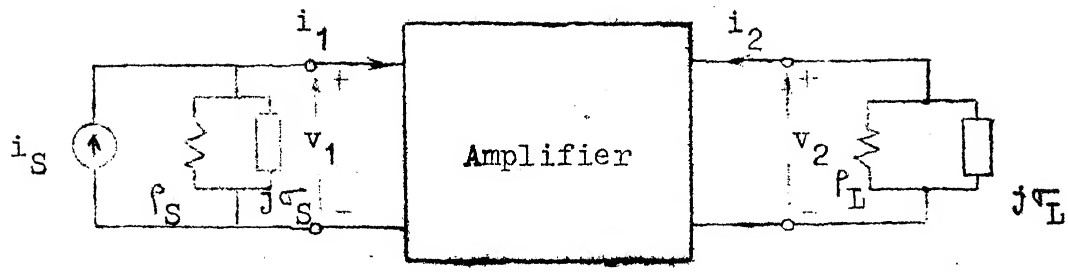


Fig. 1.1

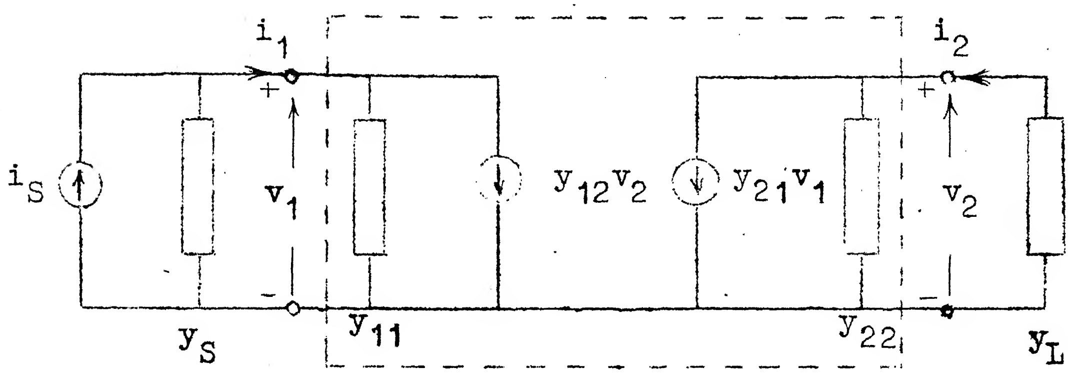


Fig. 1.2

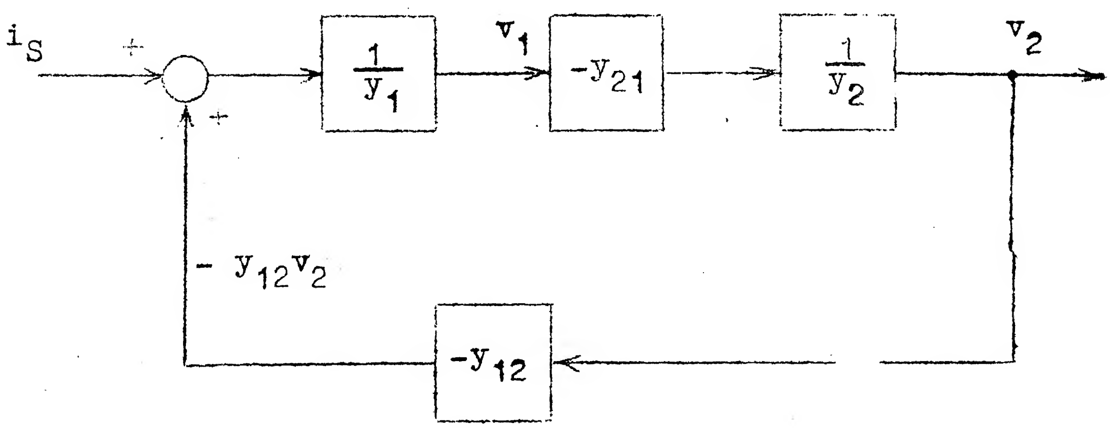


Fig. 1.3

Therefore, Eqns. (1.1) become

$$\left.\begin{aligned} i_S &= y_1 v_1 + y_{12} v_2 \\ -\frac{y_{21}}{y_2} v_1 &= v_2 \end{aligned}\right\} \qquad \ldots \quad (1.3)$$

where

$$y_1 = y_{11} + y_S \quad \text{and} \quad y_2 = y_{22} + y_L \qquad \ldots \quad (1.4)$$

Equations (1.3) may be represented by a block diagram as in Fig. 1.3. This block diagram shows y_{12} as a feedback term, and the loop gain is given by

$$g_1 = \frac{y_{12}\, y_{21}}{y_1\, y_2} \qquad \ldots \quad (1.5)$$

A two-port network in h-environment, that is, with a series source termination and a shunt load termination as shown in Fig. 1.4, is easier to analyse with the two-port represented by its h-equivalent circuit as in Fig. 1.5. Then

$$\left.\begin{aligned} v_1 &= h_{11} i_1 + h_{12} v_2 \\ i_2 &= h_{21} i_1 + h_{22} v_2 \end{aligned}\right\} \qquad \ldots \quad (1.6)$$

Using

$$v_S = h_S i_1 + v_1 \quad \text{and} \quad i_2 = -h_L v_2 \qquad \ldots \quad (1.7)$$

gives

$$\left.\begin{aligned} v_S &= h_1 i_1 + h_{12} v_2 \\ -\frac{h_{21}}{h_2} i_1 &= v_2 \end{aligned}\right\} \qquad \ldots \quad (1.8)$$

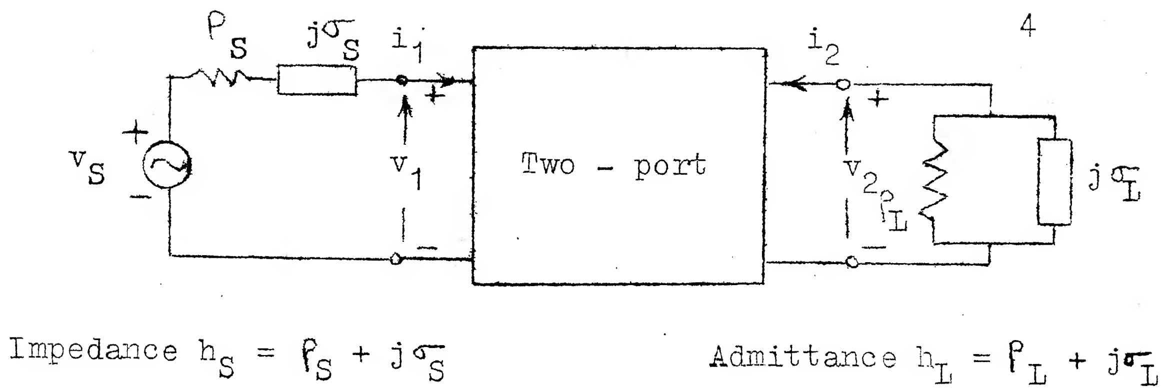


Fig. 1.4

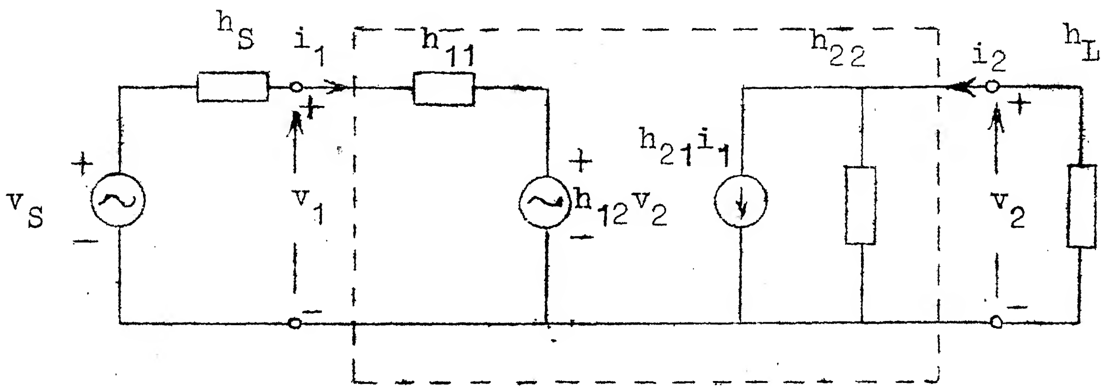


Fig. 1.5

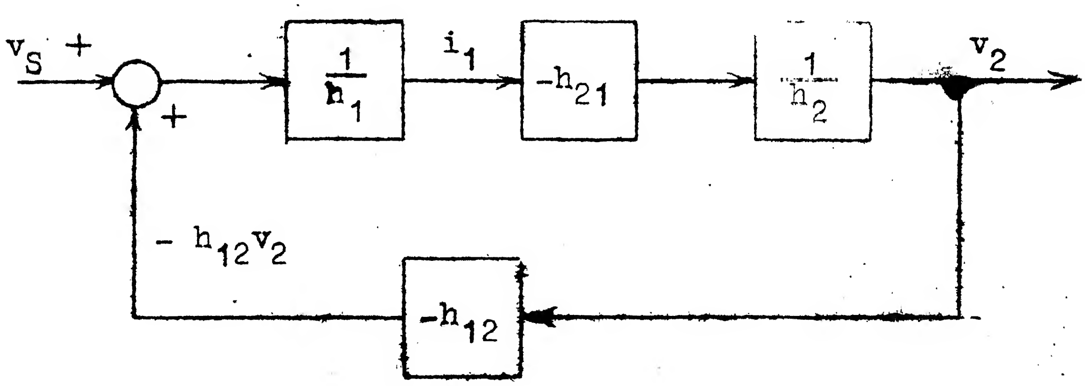


Fig. 1.6

where

$$h_1 = h_{11} + h_S \quad \text{and} \quad h_2 = h_{22} + h_L \quad \ldots \quad (1.9)$$

Eqns. (1.8) may be represented by the block diagram in Fig. 1.6, showing that the loop gain is

$$g_1 = \frac{h_{12}\, h_{21}}{h_1\, h_2} \quad \ldots \quad (1.10)$$

From Eqns. (1.5) and (1.10), it is seen that the loop gain may, in general, be represented by

$$g_1 = \frac{p_{12}\, p_{21}}{p_1\, p_2} \quad \ldots \quad (1.11)$$

where $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ represents the h-, y-, z-, or the g-matrix of the two-port, and,

$$p_1 = p_{11} + p_S \quad \text{and} \quad p_2 = p_{22} + p_L \quad \ldots \quad (1.12)$$

p_S, p_L are the source and load immittances respectively (admittance or impedance depending upon the matrix). The loop gain g_1, is generally complex.

1.2 Stability Techniques:

The internal feedback in the device may be degenerative or regenerative depending upon the loop gain g_1. As the matrix parameters p_{11}, p_{12} etc. are functions of frequency, the loop gain depends upon the frequency and the source and load immittances.If the loop gain is real, i.e.

$$g_1 = K \angle 0 \quad \ldots \quad (1.13)$$

the amplifier becomes unstable and starts oscillating, provided $K \geqslant 1$. The remedy lies in reducing the magnitude of K, so that, when real, the magnitude of loop gain is below unity.

In general, there are two methods of reducing the loop gain. In one method, known as unilateralization, the value of the total reverse parameter p_{12} is made zero. In the second method known as mismatch, the values of ρ_1 and ρ_2 are increased. In the former method, an external feedback network, usually containing both reactance and resistance elements and a transformer, is used. This feedback network is so chosen that the composite network is 'unilateral' i.e. its reverse parameter is zero. Thus any signal at the output port does not affect the input port. There are four[1] possible ways of arranging the feedback network, as shown in Figs. 1.7. After unilateralization, the composite network is conjugate matched at both ports to give maximum available power gain. The design for optimum gain involves double maximization - the composite network is conjugate matched for maximum power gain, while at the same time the feedback network is optimized to attain the largest possible value of the maximum available power gain.

The second method viz. mismatch, may be explained as follows. For a stable amplifier, maximum power gain is attained when it is conjugate matched, (See Fig. 1.8), that is,

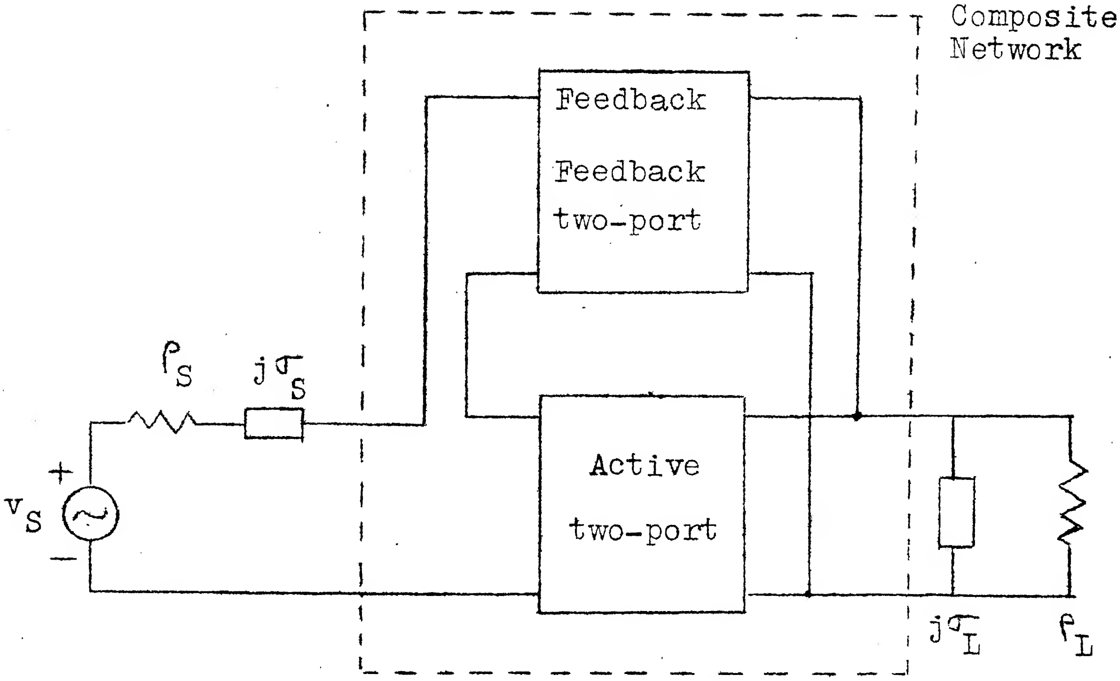


Fig. 1.7(a)

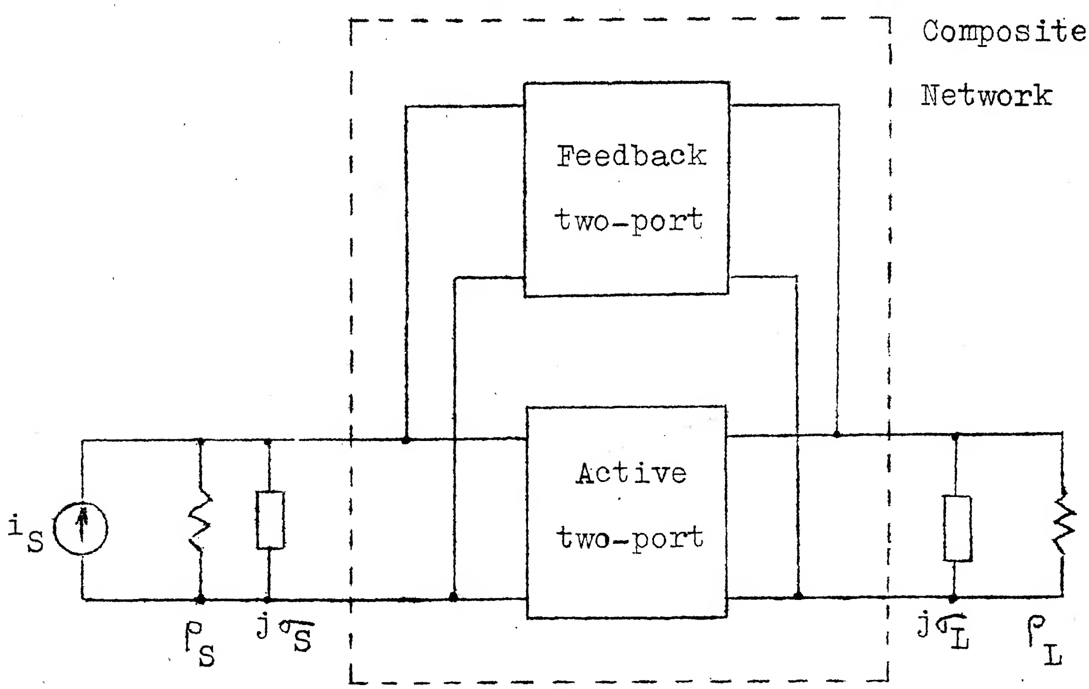


Fig. 1.7(b)

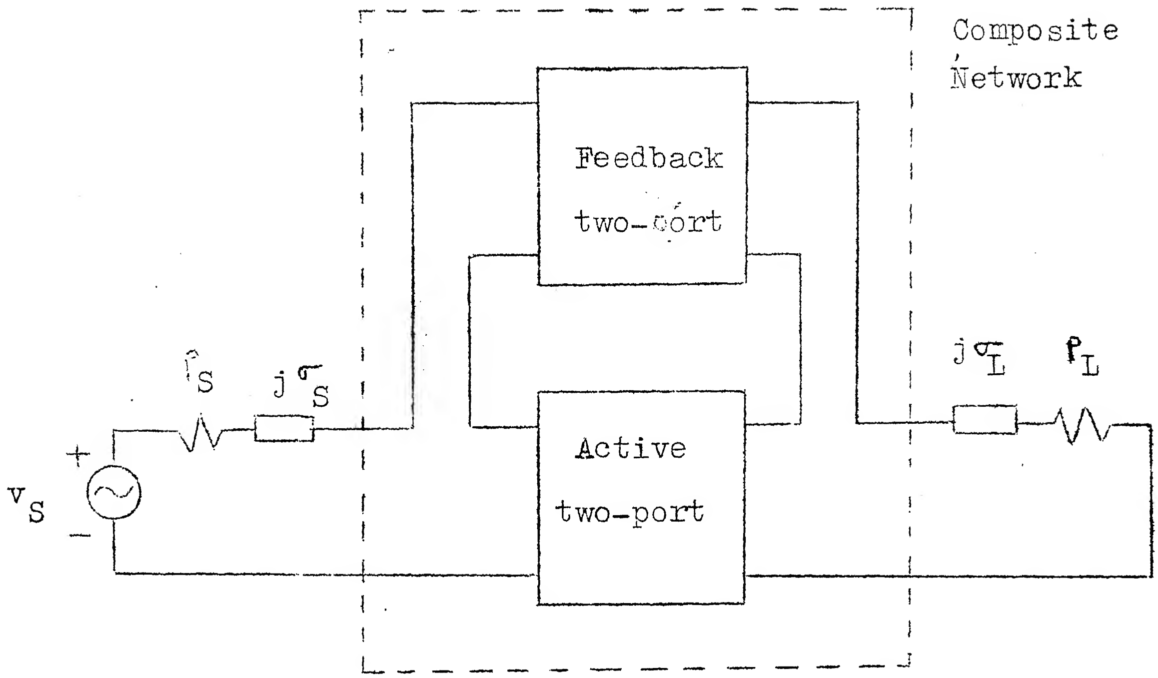


Fig. 1.7(c)

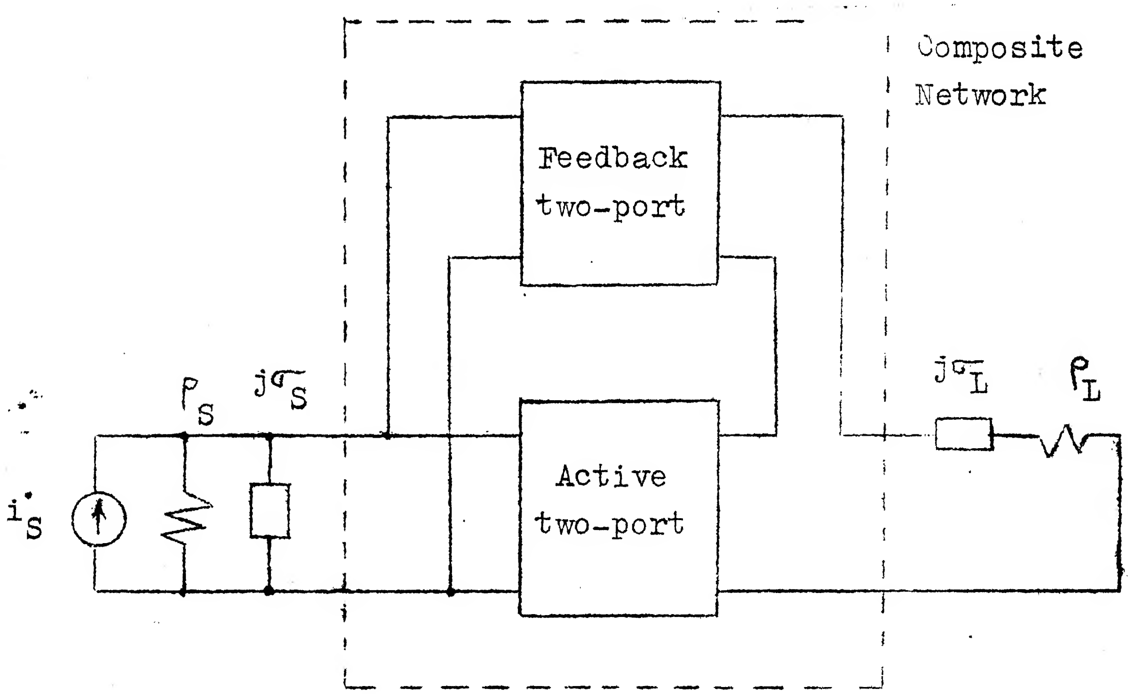


Fig. 1.7(d)

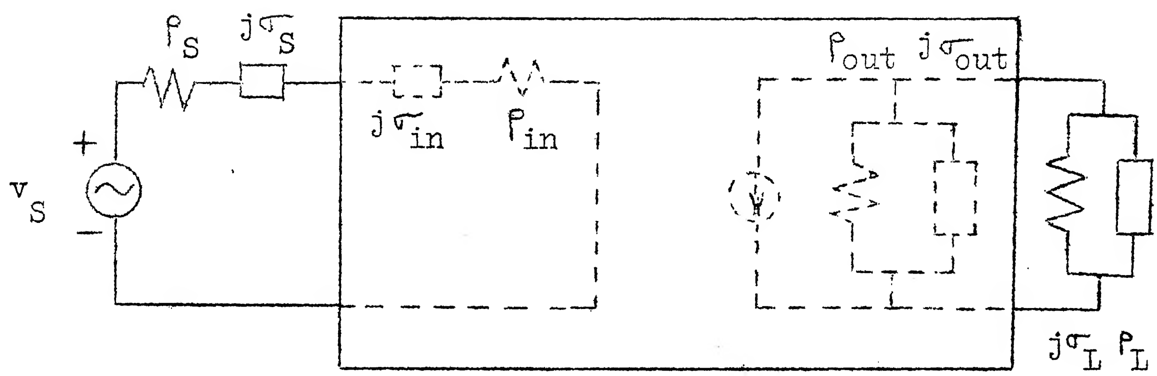

ρ_S
$j\sigma_S$
ρ_{out}
$j\sigma_{out}$
$j\sigma_{in}$
ρ_{in}
+
v_S
−
$j\sigma_L$
ρ_L

Fig. 1.8

$$\rho_S = \rho_{in} \; ; \quad \rho_L = \rho_{out}$$

and $$\sigma_S = -\sigma_{in} \; ; \quad \sigma_L = -\sigma_{out} \qquad \ldots \qquad (1.14)$$

For an unstable amplifier on the other hand, the maximum power gain is not a finite quantity. The amplifier may then be designed for a maximum operating power gain, with a given degree of stability. The maximum operating power gain is achieved with the ratios $\frac{\rho_S}{\rho_{in}}$ and $\frac{\rho_L}{\rho_{out}}$ different from unity (unlike Eqns. 1.14). Hence the name resistive mismatch or simply mismatch.

1.3 Unilateralization vs. Mismatching:

Unilateralization has three serious disadvantages. First, for this method to be useful, the unilateralization network must verv nearly eliminate the internal feedback. The internal feedback, even for a given type of transistor, varies from transistor to transistor, due to the spread in transistor parameters. Hence the external feedback network must be tailored for each transistor individually. Second, unilateralization circuits often employ a transformer for phase inversion. These transformers limit the unilateralized bandwidth. Even if the transformer is avoided it does not lead to an improvement in the bandwidth, for, the finite passive network can only roughly unilateralize the transistor over a wide band. For this reason, unilateralization techniques are usually limited to narrow-band, high gain applications. A third disadvantage is that, stability of the

amplifier is not guaranteed. If the transistor parameters vary, due to aging say, the external feedback may no longer cancel the internal feedback and in some situations may actually aid it, resulting in oscillations.

The mismatching design technique, on the other hand, has three important advantages. First, stability may be assured for all trahsistors of a particular type. Second, the input impedance can be made essentially independent of the load impedance, permitting practical multistage amplifier alignment. Third, the design technique is particularly useful in broad band techniques.

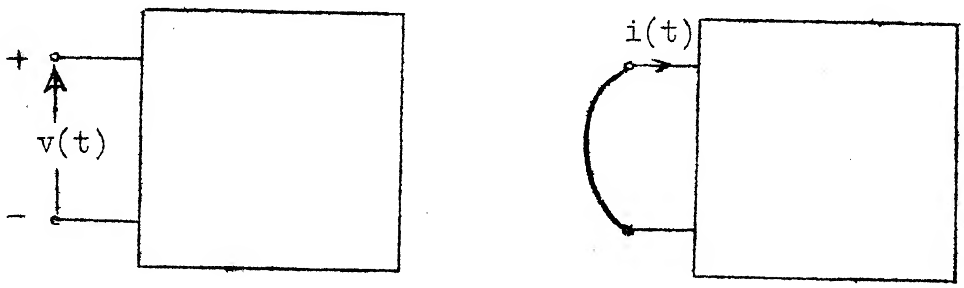
CHAPTER 2

STABILITY CONCEPTS

2.1 Stability in Natural Modes:

2.1.1 One-port Network:

Consider a linear one-port network in open and short-circuit modes as shown in Figs. 2.1. The network is said to be stable if the voltage v(t) and the current i(t) tend to zero for large time. The network is unstable if either or both the variables tend to infinity or become sinusoidal at large time. Suppose the voltage or current fail to tend to zero, when an



(a) (b)

Fig. 2.1

external source is connected to the one-port network. In such a case, the network cannot be considered unstable, as the voltage or current is controllable. Thus instability refers to uncontrolled variation (sinusoidal or tending to infinity) of the variable at large time.

The instability of the one-port is directly related to the poles and zeros of its impedance $Z(s)$. It can be shown[2] that for stability of a one-port, the poles and zeros of $Z(s)$ (or $Y(s)$) must lie entirely in the left hand complex plane (LHP).

2.1.2 Two-port Network:

Just as a one-port has two natural modes of operation, a 3-terminal (two-port) network has 5 natural modes[2] of operation as shown in Fig. 2.2. The network is stable when $v(t)$ and $i(t)$ in Figs. 2.2(a) to 2.2(c), any two of the voltages in Fig. 2.2(d) and any two of the currents in Fig. 2.2(e) tend to zero for large time. The two-port is unstable if any of the variables in any of the 5 modes becomes sinusoidal or tends to infinity at large time.

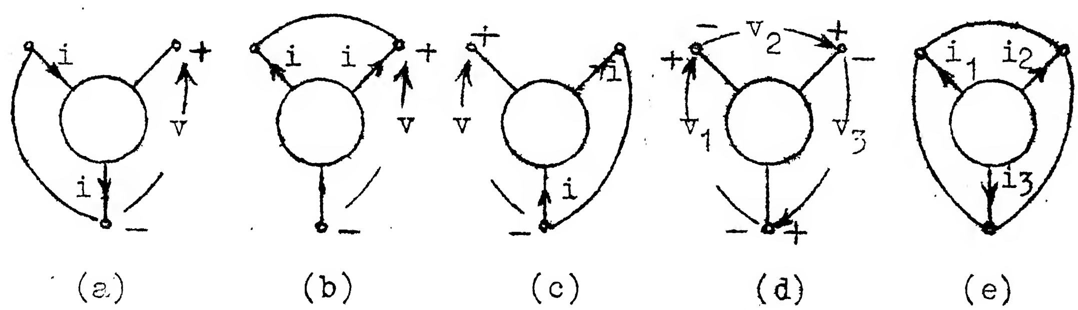


Fig. 2.2

Once again, the stability is directly related to the two-port parameters. It can be shown[2] that, for stability, the poles and zeros of p_{11} and p_{22} and the poles of p_{12} and p_{21} must lie entirely in LHP. Here 'p-' represents h-, y-, z-, or g-matrix

parameters in each of the 3 possible configurations of the 3-terminal network.

2.2 Potential Instability and Absolute Stability:

2.2.1 One-port Network:

Consider a passive termination of impedance $Z_T(s)$ connected to a one-port network as shown in Fig. 2.4. As mentioned in Section 2.1.1, the stability of one-port implies stability in both SC and OC modes, which correspond to the two extreme cases, $Z_T(s) = 0$ and $Z_T(s) = \infty$. What about the values of $Z_T(s)$ in

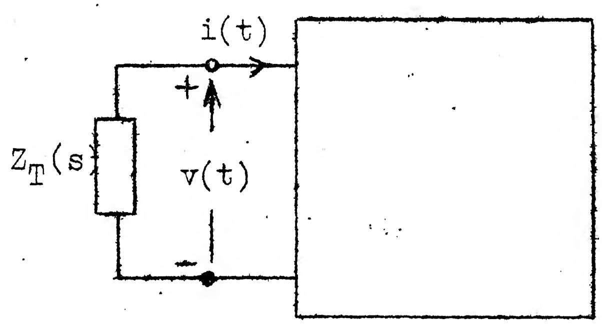


Fig. 2.3

between these two limits? The question is whether i(t) and v(t) tend to zero for large time, that is, whether the one-port is stable for an **arbitrary** value of $Z_T(s)$. If the network is indeed stable for all passive $Z_T(s)$, it is said to be absolutely stable. If a passive termination exists for which the network becomes unstable, the one-port is said to be potentially unstable,

meaning that it is stable in the two extreme cases of $Z_T(s) = 0$ and $Z_T(s) = \infty$, but unstable for some intermediate value (or values) of $Z_T(s)$.

It can be shown[2] that a passive one-port network is always absolutely stable and an active one-port network is always unstable or potentially unstable. Thus, if a one-port network is to be potentially unstable, it must be an active network and hence must violate at least one of two conditions[2] for passivity viz.,

a) The poles and zeros of p(s) must lie in LHP
b) $\mathrm{Re}\left[p(j\omega)\right] > 0$ for all $\omega \geq 0$ (2.1)

where, $p(s) = Z(s)$ or $Y(s)$, the impedance or admittance of the one-port respectively.

2.2.2 Two-port Network:

As mentioned in the preceding section, stability of a two-port implies stability in all the 5-natural modes (Figs. 2.2). Now consider a stable two-port with impedances $Z_S(s)$ and $Z_L(s)$ connected at the input and output ports as shown in Fig. 2.4. The figure shows a particular configuration with terminal 3 as the common terminal between the two ports. When $Z_S(s)$ and $Z_L(s)$ take the values 0 and ∞, the two-port finds itself in 4 of the natural modes (Figs. 2.2). Since the two-port is stable, it means

that it is stable in all these 4 modes or the 4 extreme combinations of $Z_S(s)$ and $Z_L(s)$. If, in addition, the two-port is stable for all intermediate values of $Z_S(s)$ and $Z_L(s)$, it is said to be absolutely stable. The two-port is said to be

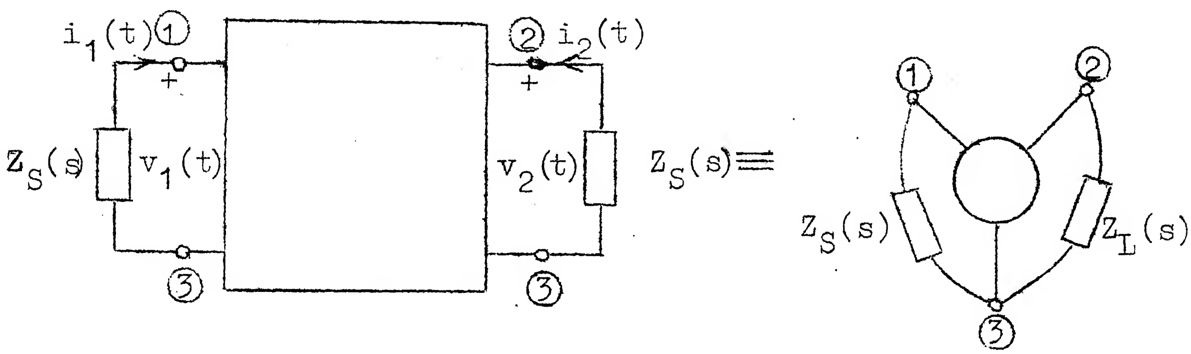


Fig. 2.4

potentially unstable if it is unstable for any intermediate combination of $Z_S(s)$ and $Z_L(s)$. Absolute stability is defined for a given configuration of the 3-terminal network. There are 3 possible configurations with a different common terminal in each and a two-port may be absolutely stable in one configuration, while potentially unstable in another. The conditions for absolute stability in terms of two-port matrix parameters are derived below.

Let a two-port be characterized by a generalized parameter matrix $\begin{bmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{bmatrix}$ which represents h-, y-, z-, or g-matrix.

Let an immittance p_L be connected at the output port. Here p_L represents an impedance or admittance depending upon the matrix chosen, as shown in Table 2.1. The resulting one-port network of Fig. 2.5(a) is tested for passivity using conditions (2.1). This is repeated for the one-port network of Fig. 2.5(b). The two-port is stable, if both the one-ports as shown in Figs. 2.5 are passive i.e. if $p_{in}(s)$ and $p_{out}(s)$ satisfy conditions (2.1). Now, it can be shown[2] that, condition (2.1)a is satisfied if the two-port is stable to start with i.e. stable in all the 5 natural modes. Since only a stable two-port is being considered. condition (2.1)b alone need be investigated. Then

$$\mathrm{Re}\left[p_{in}(j\omega)\right] = \rho_{in} = \mathrm{Re}\left[p_{11}(j\omega) - \frac{p_{12}(j\omega)\, p_{21}(j\omega)}{p_{22}(j\omega) + p_L(j\omega)}\right] > 0$$

and $$\mathrm{Re}\left[p_{out}(j\omega)\right] = \rho_{out} = \mathrm{Re}\left[p_{22}(j\omega) - \frac{p_{12}(j\omega)\, p_{21}(j\omega)}{p_{11}(j\omega) + p_S(j\omega)}\right] > 0$$

$$\text{for all } \omega \geqslant 0 \qquad \ldots \qquad (2.2)$$

must be satisfied for all $p_L(j\omega)$ and $p_S(j\omega)$ respectively. Here p_{in} (or p_{out}) is an impedance or admittance depending upon the matrix as shown in Table 2.1. Two special cases are those of $p_S(j\omega) = \infty$ and $p_L(j\omega) = \infty$. Each case corresponds to a short-circuit or an open-circuit depending upon the matrix used. Then conditions (2.2) give,

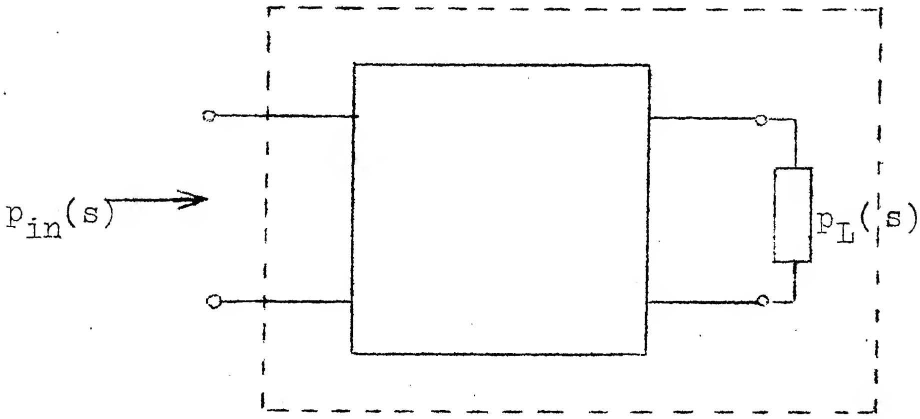


Fig. 2.5(a)

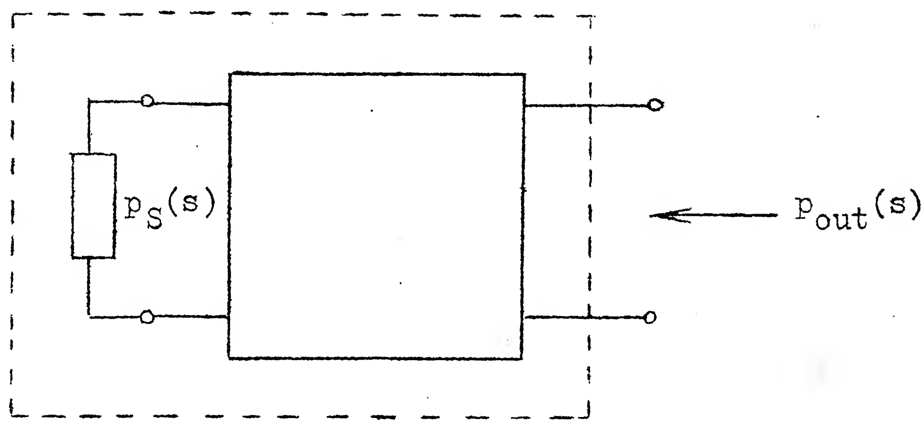


Fig. 2.5(b)

Table 2.1

Matrix	z	y	h	g
p_{11}, p_S and p_{in}	Imp.	Adm.	Imp.	Adm.
p_{22}, p_L and p_{out}	Imp.	Adm.	Adm.	Imp.

$$\left.\begin{array}{ll} & \mathrm{Re}\,[p_{11}(j\omega)] \text{ or } \rho_{11}(j\omega) > 0 \\ \text{and} & \mathrm{Re}\,[p_{22}(j\omega)] \text{ or } \rho_{22}(j\omega) > 0 \end{array}\right\} \text{ for all } \omega \geq 0 \quad ..(2.3)$$

2.3 Conditions for Absolute Stability: Stern's[3] Proof:

From Eqn. (2.2),

$$\rho_{in} = \mathrm{Re}\left[p_{11} - \frac{p_{12}\, p_{21}}{p_{22} + p_L} \right] \quad \ldots (2.4)$$

$$= \mathrm{Re}\left[\rho_{11} + j\,\sigma_{11} - \frac{M + jN}{(\rho_{22} + \rho_L) + j\,(\sigma_{22} + \sigma_L)} \right] \quad ..(2.5)$$

where

$$p_{12}\, p_{21} = M + jN = L \underline{/\theta} \quad \ldots (2.6)$$

$$p_{11} = \rho_{11} + j\,\sigma_{11} \;;\; p_{22} = \rho_{22} + j\,\sigma_{22} \quad \ldots (2.7)$$

and

$$p_L = \rho_L + j\sigma_L \quad \ldots (2.8)$$

Therefore,

$$\rho_{in} = \rho_{11} - \frac{M(\rho_{22} + \rho_L) + N(\sigma_{22} + \sigma_L)}{(\rho_{22} + \rho_L)^2 + (\sigma_{22} + \sigma_L)^2} \quad \ldots (2.9)$$

$$= \frac{(\rho_{22}+\rho_L)^2 + (\sigma_{22}+\sigma_L)^2 - \frac{M}{\rho_{11}}(\rho_{22}+\rho_L) - \frac{N}{\rho_{11}}(\sigma_{22}+\sigma_L)}{\frac{1}{\rho_{11}}\left[(\rho_{22}+\rho_L)^2 + (\sigma_{22}+\sigma_L)^2\right]} \quad (2.10)$$

$$= \frac{Nr}{Dr} \quad \ldots (2.11)$$

Only the sign of ρ_{in} is of interest, and as Dr is positive ($\rho_{11} > 0$, from Eqn. 2.2) only the sign of Nr need be considered.

$$Nr = (\rho_{22}+\rho_L)^2 + (\sigma_{22}+\sigma_L)^2 - \frac{M}{\rho_{11}}(\rho_{22}+\rho_L) - \frac{N}{\rho_{11}}(\sigma_{22}+\sigma_L) \quad \ldots\ldots(2.12)$$

Therefore,

$$Nr + \frac{L^2}{4\rho_{11}^2} = \left\{\rho_L + \left(\rho_{22} - \frac{M}{2\rho_{11}}\right)\right\}^2 + \left\{\sigma_L + \left(\sigma_{22} - \frac{N}{2\rho_{11}}\right)\right\}^2 \quad \ldots\ldots(2.13)$$

after making use of Eqn. (2.6). Equation (2.13) represents a family of concentric circles with Nr as the parameter, with a common centre $\left\{\left(\frac{M}{2\rho_{11}} - \rho_{22}\right), \left(\frac{N}{2\rho_{11}} - \sigma_{22}\right)\right\}$ in the (ρ_L, σ_L) plane as shown in Fig. 2.6(a). The radius varies from 0 to ∞, as Nr increases from $-\frac{L^2}{4\rho_{11}^2}$ to ∞, The critical circle i.e.,

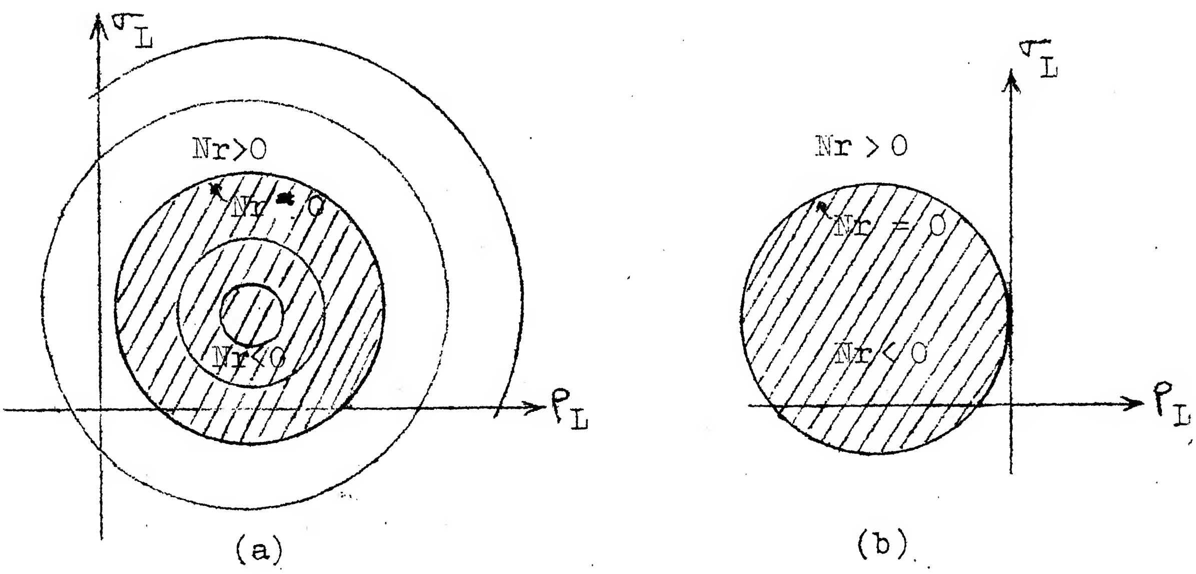


Fig. 2.6

the circle for Nr = 0 is marked in the Figure. All the inner circles correspond to Nr < 0 and the outer circles to Nr > 0.

If Nr is to remain positive for any passive termination $p_L = \rho_L + j\sigma_L$ which can be represented by a point (ρ_L, σ_L) in the right half plane (ρ_L cannot be negative) then, every point in the right half plane including the σ_L - axis must be lie outside the critical circle. In other words, the critical circle must be entirely in the left half plane. Therefore,

-(x co-ordinate of the centre) > radius of critical circle

or
$$\rho_{22} - \frac{M}{2\rho_{11}} > \frac{L}{2\rho_{11}} \qquad \ldots \quad (2.14)$$

or
$$2\rho_{11}\rho_{22} > L + M \qquad \ldots \quad (2.15)$$

for all $\omega \geqslant 0$

Rewriting the conditions for absolute stability from Eqns. (2.3) and (2.15),

$$\left.\begin{aligned} &\rho_{11} > 0, \quad \rho_{22} > 0 \qquad &\ldots \quad (2.16)\\ &2\rho_{11}\rho_{22} > L + M \qquad &\ldots \quad (2.17)\end{aligned}\right\}$$

for all $\omega \geqslant 0$

These are usually referred to as Llewellyn's[4] stability criteria for two-port network. If resistive terminations ρ_S and ρ_L are added and are considered part of a modified network, condition (2.17) becomes,

$$2\rho_1\rho_2 > L + M \qquad \ldots \quad (2.18)$$

for all $\omega \geqslant 0$

where
$$\rho_1 = \rho_{11} + \rho_S \qquad \ldots \quad (2.19)$$
$$\rho_2 = \rho_{22} + \rho_L \qquad \ldots \quad (2.20)$$

The addition of ρ_S and ρ_L is shown in Fig. 2.7 for the case in which ρ_S is a resistance and ρ_L a conductance (h-matrix).

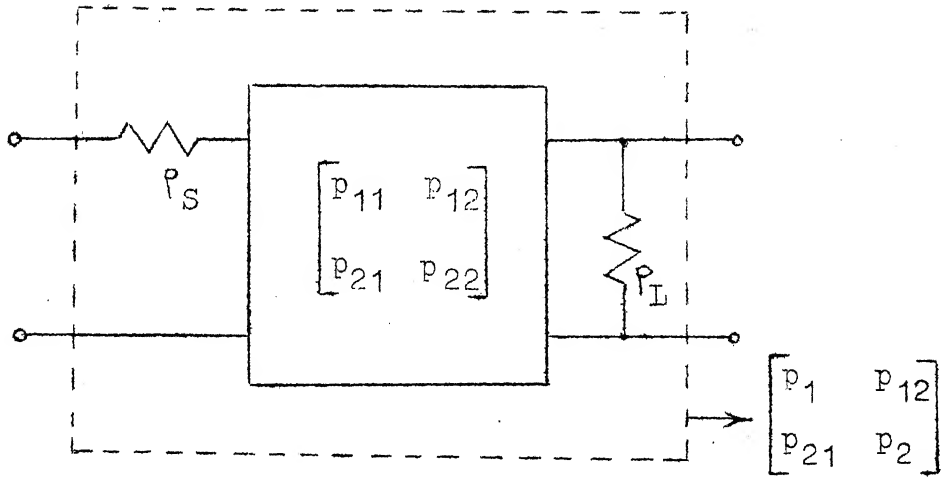


Fig. 2.7

If one starts with Eqn. (2.2)b instead of Eqn. (2.2)a, the same result viz., condition (2.15) is obtained.

2.4 Various Stability Factors:

The absolute stability criteria of Eqns. (2.15) and (2.18) are rewritten below for convenience:

$$2\rho_{11}\rho_{22} > L + M \qquad \ldots \quad (2.21)$$

$$2\rho_1\rho_2 > L + M \qquad \ldots \quad (2.22)$$

Stern[5] defined in 1957, a stability factor k as

$$k = \frac{2\rho_1\rho_2}{L + M} \qquad \ldots \quad (2.23)$$

Thus $k > 1$ for absolute stability. This is a stability factor of the modified network shown in Fig. 2.7. An inherent stability factor k_i is defined by

$$k_i = \frac{2\rho_{11}\rho_{22}}{L + M} \qquad \ldots \quad (2.24)$$

Since $\rho_1 \geqslant \rho_{11}$ and $\rho_2 \geqslant \rho_{22}$, therefore,

$$k \geqslant k_i \qquad \ldots \quad (2.25)$$

Thus a two-port network which is potentially unstable ($k_i \leqslant 1$) may be modified suitably so that the modified network is absolutely stable ($k > 1$). The magnitude of the stability factor k (or k_i) depends upon the matrix used, and may be denoted by k_p where $p \equiv h, y, z$ or g. The magnitudes k_p are all > 1 or all $= 1$ or all < 1 in the regions of absolute stability, marginal stability and potential instability respectively. Except in the case of marginal stability the values of k_p are different. For a given matrix the value of k is, in general, different for the 3 different configurations. The stability factor k may be < 1 in one configuration and > 1 in another, which means that the two-port is potentially unstable in the former configuration and absolutely stable in the latter. But in a given configuration, k (or k_i) is independent of as to which port is chosen as input port.

Venkateswaran[1] and Boothroyd define the performance factor n, and the inherent performance factor n_i by

$$n = \frac{\rho_1 \rho_2}{L} \qquad \text{and} \qquad n_i = \frac{\rho_{11} \rho_{22}}{L} \qquad \ldots \quad (2.26)$$

From Eqns. (2.23) and (2.24) the following relations are obtained:

$$n = \frac{\rho_1 \rho_2}{L} = k \frac{L + M}{2L} = k \, \Pi \qquad \ldots \quad (2.27)$$

and $$n_i = \frac{\rho_{11} \rho_{22}}{L} = k_i \frac{L + M}{2L} = k_i \, \Pi \qquad \ldots \quad (2.28)$$

where $$\Pi = \frac{L + M}{2L} = \frac{1 + \cos\theta}{2} \qquad \ldots \quad (2.29)$$

using Eqn. (2.6)

Thus $n_i > \Pi$ for absolute stability of a two-port network or $n > \Pi$ for absolute stability of a modified network. The performance factor n was proposed because the maximum operating power gain of a two-port network with $k\Pi \gg 1$, which implies high stability, can be simply expressed in terms of n.

Another pair of factors called invariant factor η and inherent invariant factor η_i are defined as follows:

$$\eta = \frac{2\rho_1 \rho_2 - M}{L} \qquad \ldots \quad (2.30)$$

$$\eta_i = \frac{2\rho_{11} \rho_{22} - M}{L} \qquad \ldots \quad (2.31)$$

Unlike k and n, the invariant factor η is independent of the matrix used.

Venkateswaran[7] defines an invariant stability factor S by

$$S = \eta + \sqrt{\eta^2 - 1} \qquad \ldots \quad (2.32)$$

Similarly, the inherent invariant stability factor S_i is defined by

$$S_i = \eta_i + \sqrt{\eta_i^2 - 1} \qquad \ldots \quad (2.33)$$

For absolute stability of the original and modified network respectively,

$$S_i > 1 \qquad \ldots \quad (2.34)$$

and
$$S > 1 \qquad \ldots \quad (2.35)$$

When the two-port network is potentially unstable, corresponding to $\eta_i < 1$, S_i becomes complex. The stability factor S_i was proposed because it is simply related to the MAG (maximum available power gain) of an absolutely stable two-port network. As S_i (or S) is a function of η_i (or η) alone it too is independent of the matrix used.

Venkateswaran[8] proposed in 1968 an invariant alignability factor δ which is defined as

$$\delta = \frac{2\rho_1 \rho_2 - M - L}{2L} = \frac{\eta - 1}{2} \qquad \ldots \quad (2.36)$$

$$= n - \Pi \qquad \ldots \quad (2.37)$$

For absolute stability, $n > \Pi$ or $\delta > 0$. This stability based factor was proposed from the consideration of sensitivity of total port immittance ($p_S + p_{in}$ or $p_L + p_{out}$) at one port to a small change in termination (p_L or p_S) at the other port.

Callendar[9] proposed in 1968 two more stability factors. But they have been shown[10] to be simply related to Venkateswaran and Boothroyd's performance factor n proposed in 1960.

The inequalities to be satisfied by the various stability based factors for the absolute stability of a two-port are

$$k_i > 1;\ n_i > \Pi;\ \eta_i > 1;\ S_i > 1;\ \delta_i > 0 \qquad \ldots(2.38)$$

2.5 <u>Interrelations Between the Stability Factors</u>:

The various factors k, n, η, S and δ are all stability based factors and can be expressed in terms of one another. The relations between these factors is given in a paper[8]. Similar relationships exist between the various inherent factors k_i, n_i, η_i, S_i and δ_i. The relationships between the performance factor n and the other factors are also shown graphically in Figs. (2.8). In each figure a plane is defined by n and θ axes and k, η etc. are varied as parameters. The shaded portion in each graph corresponds to $n < \Pi$ and hence represents a potentially unstable region.

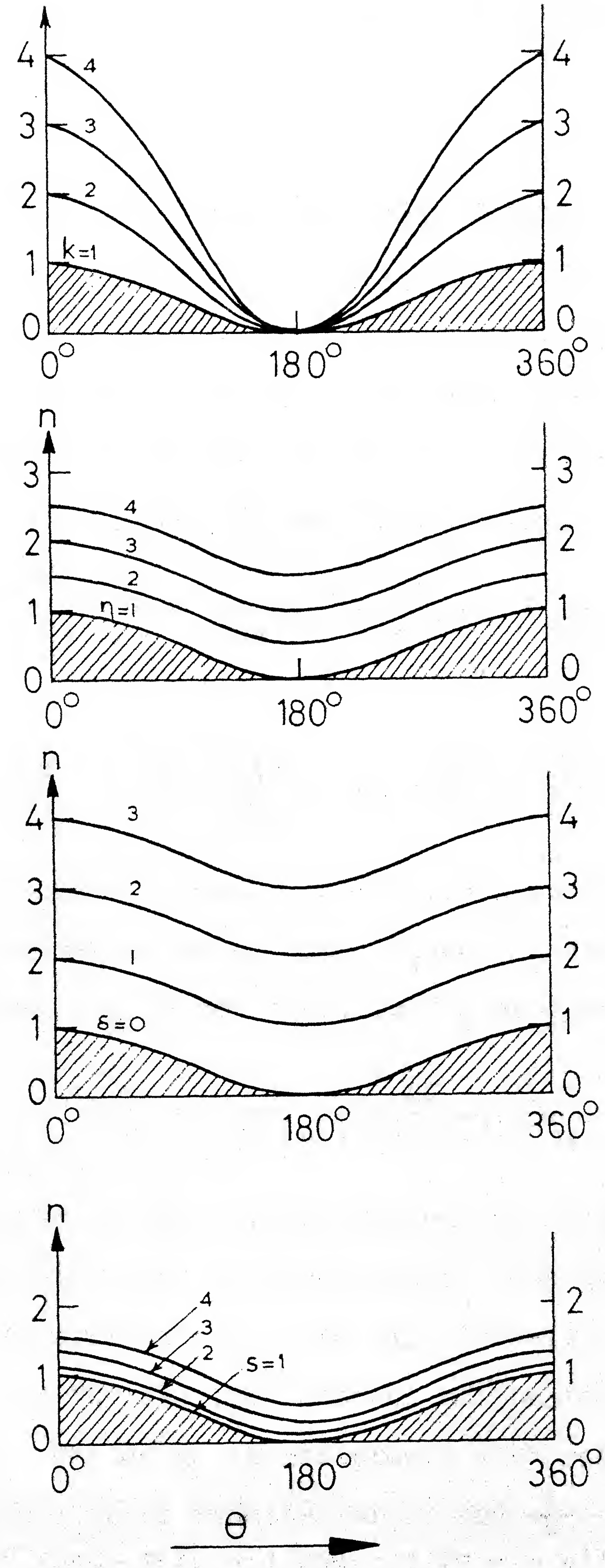


FIG. 2·8 CURVES OF CONSTANT-k, η, s AND δ in n-θ PLANE

2.6 Internal Loop Gain and Stability Factors:

It has been shown by Venkateswaran[7] that the factors k, n and S are simply related to the internal loop gain g_1 of Eqn. (1.11). Before proceeding with these factors however, it will be revealing to see how the loop gain g_1, varies with σ_S and σ_L, while ρ_S and ρ_L are held constant.

$$g_1 = \frac{p_{12}\ p_{21}}{p_1\ p_2} = \frac{p_{12}\ p_{21}}{(\rho_1+j\sigma_1)(\rho_2+j\sigma_2)} = \frac{L\ \underline{/\theta}}{\rho_1\rho_2(1+j\lambda_1)(1+j\lambda_2)} \quad \ldots \quad (2.39)$$

where $\lambda_1 = \dfrac{\sigma_1}{\rho_1} = \dfrac{\sigma_S+\sigma_{11}}{\rho_S+\rho_{11}}$; $\lambda_2 = \dfrac{\sigma_2}{\rho_2} = \dfrac{\sigma_L+\sigma_{22}}{\rho_L+\rho_{22}}$ (2.40)

λ_1 and λ_2 vary from $-\infty$ to $+\infty$ as σ_S and σ_L vary from $-\infty$ to $+\infty$. The variation of g_1 with λ_1 and λ_2 is plotted in Fig. 2.9. Here, λ_2 is set equal to $W\lambda_1$ so that

$$g_1 = \frac{L\ \underline{/\theta}}{\rho_1\ \rho_2\ (1+j\lambda_1)(1+jW\lambda_1)} \quad \ldots \quad (2.41)$$

Each curve in Fig. 2.9 is traced out when λ_1 varies from $-\infty$ to $+\infty$, W being used as a parameter. The family of curves results when W varies from -1 to $+1$. Actually, W should vary from $-\infty$ to $+\infty$ to take into account all possible combinations of λ_1 and λ_2. But as g_1 is reciprocal with respect to λ_1 and λ_2, the same curve results for W and $\frac{1}{W}$. Thus, a further variation of W from $-\infty$ to -1 and $+1$ to $+\infty$ will yield exactly the same curves.

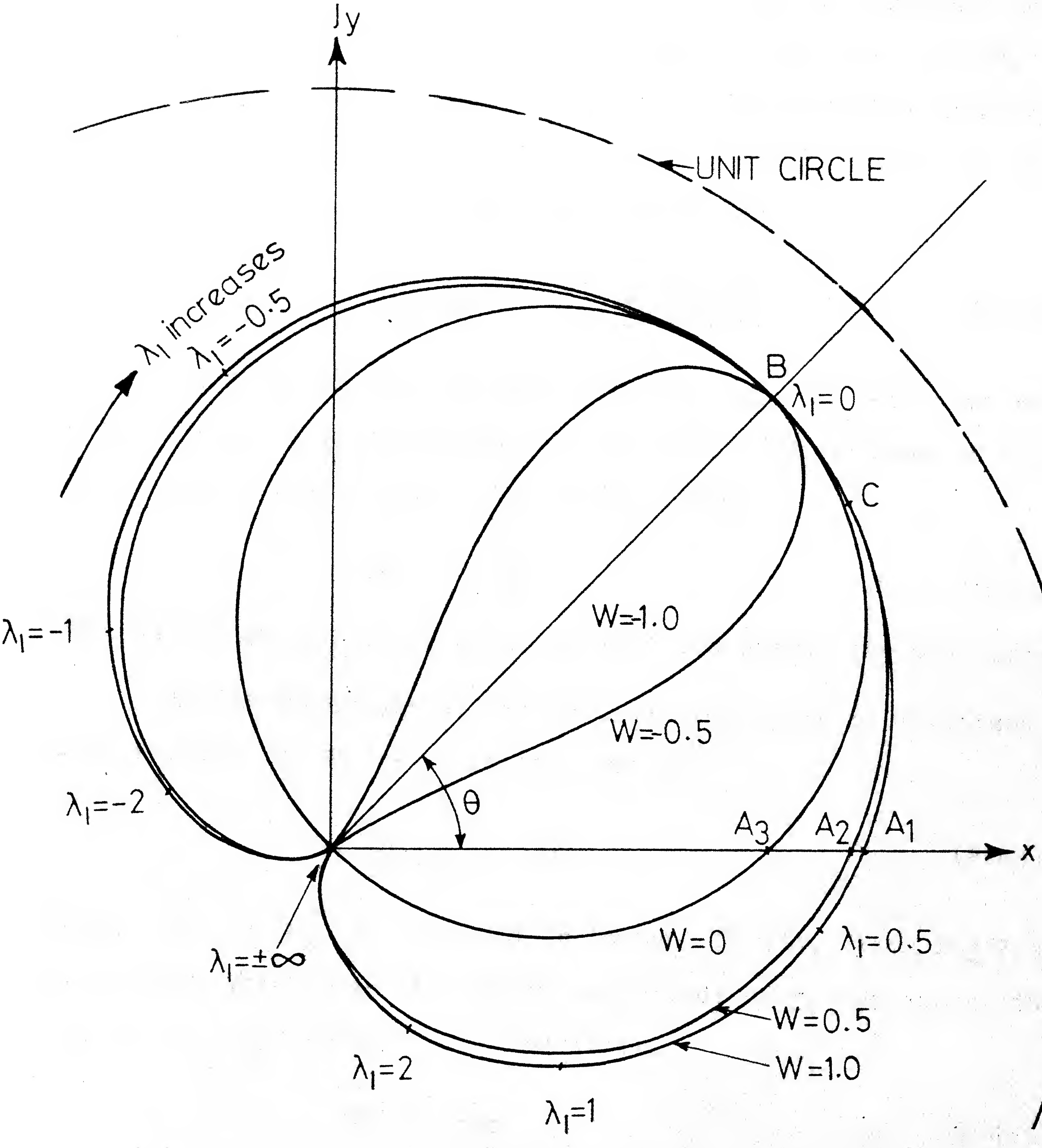


FIG. 2.9 VARIATION OF LOOP GAIN g_l WITH λ_1 AND λ_2

The interrelations between k, n, S and the internal loop gain g_1 may be understood with the help of Fig. 2.9. First, if the internal loop gain is constrained to be real by suitable choice of σ_S and σ_L and then maximized[7] with respect to the two variables, its value $g_{1r\,max}$ is given by

$$g_{1r\,max} = \frac{1}{k} = \frac{L+M}{2\rho_1\rho_2} \qquad \ldots \quad (2.42)$$

From Fig. 2.9 it is seen that OA_1, OA_2, OA_3 etc. are the values of g_1, when constrained to be real. The maximum of these, OA_1, occurs for W = 1 i.e. $\lambda_1 = \lambda_2$. Thus,

$$OA_1 = \frac{1}{k} \qquad \ldots \quad (2.43)$$

For stability, $k > 1$ and hence A_1 must lie within the unit circle.

If the magnitude of the internal loop gain is maximized with respect to σ_1, σ_2, it is given by[7]

$$|g_1|_{max} = \frac{1}{n} \qquad \ldots \quad (2.44)$$

Here, $\lambda_1 = \lambda_2 = 0$ according to Eqn. (2.41), From Fig. 2.9 it is seen that OB is the maximum magnitude of g_1 and occurs for $\lambda_1 = 0$, $\lambda_2 = W\lambda_1 = 0$. Therefore,

$$OB = \frac{1}{n} \qquad \ldots \quad (2.45)$$

It will be seen from the figure that, if $\theta = 0$, the points A_1 and B coincide, giving k = n, which is also evident from Eqns. (2.27) and (2.29).

If an absolutely stable two-port ($\eta_i > \pi$ or $S_i > 1$) is conjugate matched at both the ports the magnitude of the internal loop gain is given by [7]

$$|g_{1c}| = \frac{1}{S_i} = \frac{1}{\eta_i + \sqrt{\eta_i^2 - 1}} \qquad (2.46)$$

The loop gain g_{1c} for conjugate matched condition is complex in general and may be represented by a point C on the curve for W = 1 or $\lambda_1 = \lambda_2$ in Fig. 2.9, since conjugate matching gives $\lambda_1 = \lambda_2$. It has been shown[7] that g_{1c} is real only for $\theta = 0$. In such a case, the points A_1, B and C will coincide giving $n = k = S_i$.

CHAPTER 3

COMPUTATION OF λ_o

3.1 Maximization of Operating Power Gain:

If a two-port is absolutely stable, then conjugate matching[7,11] (i.e. $p_s = p_{in}^*$; $p_L = p_{out}^*$) gives a maximum possible power gain M.A.G., which is given by[7],

$$(\text{power gain})_{max} = (\text{operating power gain})_{max}$$
$$= (\text{available power gain})_{max}$$

$$\text{M.A.G.} = \left|\frac{p_{21}}{p_{12}}\right| \frac{1}{\eta_i + \sqrt{\eta_i^2 - 1}} = \left|\frac{p_{21}}{p_{12}}\right| \frac{1}{S_i} \quad \ldots (3.1)$$

where η_i is the inherent invariant factor of Eqn. (2.31). The terminations for conjugate matching are given by[1]

$$\rho_S = \rho_{in} = \frac{L}{2\rho_{22}} \sqrt{\eta_i^2 - 1}; \quad \rho_L = \rho_{out} = \frac{L}{2\rho_{11}} \sqrt{\eta_i^2 - 1} \quad \ldots (3.2)$$

$$\sigma_S = -\sigma_{in} = -\sigma_{11} + \frac{N}{2\rho_{22}}; \quad \sigma_L = -\sigma_{out} = -\sigma_{22} + \frac{N}{2\rho_{11}} \quad \ldots (3.3)$$

If the two-port is potentially unstable i.e. $\eta_i < 1$ or $n_i < \Pi$ etc., conjugate matching is not possible as seen from Eqns. (3.2). Hence a maximum power gain or a maximum available power gain does not exist i.e., it is not finite. A finite operating power gain, however, exists provided, the terminations are so constrained that, the loop gain $g_1 = \frac{p_{12}\, p_{21}}{p_1\, p_2}$, when real

and maximized is < 1. This implies that the stability factor $k = \frac{2\rho_1 \rho_2}{L+M} > 1$ or $n = \frac{\rho_1 \rho_2}{L} > \prod$ etc.

The operating power gain may be maximized[1] for a given value of Venkateswaran and Boothroyd's performance factor n. The operating power gain of a two port is given by

$$g_p = \frac{4\,|p_{21}|^2 \rho_S \rho_L}{|p_1 p_2 - p_{12} p_{21}|^2} = \frac{4\,|p_{21}|^2 \rho_S \rho_L}{|(\rho_1 + j\sigma_1)(\rho_2 + j\sigma_2) - (M + jN)|^2} \quad \ldots (3.4)$$

$$= \frac{4\,|p_{21}|^2 \rho_S \rho_L}{|\rho_1 \rho_2 (1 + j\lambda_1)(1 + j\lambda_2) - (M + jN)|^2} \quad \ldots (3.5)$$

where

$$\lambda_1 = \frac{\sigma_1}{\rho_1} = \frac{\sigma_{11} + \sigma_S}{\rho_{11} + \rho_S} \;;\quad \lambda_2 = \frac{\sigma_2}{\rho_2} = \frac{\sigma_{22} + \sigma_L}{\rho_{22} + \rho_L} \quad \ldots (3.6)$$

$$\therefore\; g_p = \frac{4|p_{21}|^2 \rho_S \rho_L}{L^2 \left| \frac{\rho_1 \rho_2}{L} (1 + j\lambda_1)(1 + j\lambda_2) - (\cos\theta + j\sin\theta) \right|^2} \quad \ldots (3.7)$$

$$= \frac{4 \left|\frac{p_{21}}{p_{12}}\right| \frac{1}{L} \rho_S \rho_L}{|n(1 + j\lambda_1)(1 + j\lambda_2) - (\cos\theta + j\sin\theta)|^2} \quad \text{(using Eqn. 2.6)} \quad \ldots (3.8)$$

$$= \frac{Nr}{Dr} \quad \ldots (3.9)$$

The gain g_p may be treated as a function of ρ_S, ρ_L, λ_1 and λ_2 rather than of ρ_S, ρ_L, σ_S and σ_L. The gain g_p is to be maximized with respect to these 4 variables, but subject to overall stability of the two-port i.e.,

$$\frac{(\rho_{11} + \rho_S)(\rho_{22} + \rho_L)}{L} = \frac{\rho_1 \rho_2}{L} = n \qquad \ldots \quad (3.10)$$

where, n is chosen to be $> \pi$ and n_i. The inherent performance factor n_i is given by

$$\frac{\rho_{11} \rho_{22}}{L} = n_i \qquad \ldots \quad (3.11)$$

The denominator Dr may now be minimized. Since it is reciprocal with respect to λ_1 and λ_2, the minimum will occur when $\lambda_1 = \lambda_2 = \lambda$. Therefore,

$$Dr = |n(1 + j\lambda)^2 - \cos\theta - j\sin\theta|^2 \qquad \ldots \quad (3.12)$$

$$= \left[n(1 - \lambda^2) - \cos\theta\right]^2 + \left[2n\lambda - \sin\theta\right]^2 \qquad (3.13)$$

Minimizing Dr gives,

$$Dr_{min} = \left[n(1 - \lambda_o^2) - \cos\theta\right]^2 + \left[2n\lambda_o - \sin\theta\right]^2 \quad (3.14)$$

where, λ_o is the appropriate real root of the equation

$$\lambda^3 - b\lambda - a = 0 \qquad \ldots \quad (3.15)$$

where

$$b = -\left(1 + \frac{\cos\theta}{n}\right); \qquad a = \frac{\sin\theta}{n} \qquad \ldots \quad (3.16)$$

The cubic equation may have only one real root, the other being complex conjugate. In such a case, the single real root is substituted for λ_o in Eqn. (3.14). If the cubic equation has 3 real roots, then as shown in Section 4.2, one of

the roots will be positive and the other two negative or one will be negative and the other two positive. Then as shown in Section 4.2 and Appendix A, the single positive root or the single negative root is chosen respectively and substituted for λ_o in Eqn. (3.14).

The numerator Nr may be maximized with respect to ρ_S and ρ_L as follows,

$$Nr = 4\left|\frac{p_{21}}{p_{12}}\right| \frac{1}{L}(\rho_1 - \rho_{11})(\rho_2 - \rho_{22}) \quad \ldots \quad (3.17)$$

$$= 4\left|\frac{p_{21}}{p_{12}}\right| \frac{\rho_1\rho_2}{L}\left(1 - \frac{\rho_{11}}{\rho_1}\right)\left(1 - \frac{\rho_{22}}{\rho_2}\right) \quad \ldots \quad (3.18)$$

$$= 4\left|\frac{p_{21}}{p_{12}}\right| n\left(1 - \frac{\rho_{11}}{\rho_1} - \frac{\rho_{22}}{\rho_2} + \frac{\rho_{11}\rho_{22}}{\rho_1\rho_2}\right) \quad \ldots \quad (3.19)$$

$$= 4\left|\frac{p_{21}}{p_{12}}\right| n\left(1 - \frac{\rho_{11}}{\rho_1} - \frac{\rho_{22}}{\rho_2} + \frac{n_i}{n}\right) \quad \ldots \quad (3.20)$$

(from Eqns. 3.10 and 3.11)

$\frac{dNr}{d\rho_1} = 0$ gives,

$$\frac{\rho_{11}}{\rho_1} = \frac{\rho_{11}}{\rho_{11} + \rho_S} = \sqrt{\frac{n_i}{n}} \quad \ldots \quad (3.21)$$

$$= \frac{\rho_{22}}{\rho_2} = \frac{\rho_{22}}{\rho_{22} + \rho_L} \quad \ldots \quad (3.22)$$

(using Eqns. 3.10 and 3.11)

Therefore,

$$Nr_{max} = 4\left|\frac{p_{21}}{p_{12}}\right| n\left(1 - \sqrt{\frac{n_i}{n}}\right)^2 \quad \ldots \quad (3.23)$$

The maximum operating power gain may now be written as,

$$g_{max\ n} = \frac{Nr_{max}}{Dr_{min}}$$

$$= 4 \left|\frac{p_{21}}{p_{12}}\right| \frac{n\left(1 - \sqrt{\frac{n_i}{n}}\right)^2}{\left\{\left[n(1-\lambda_o^2) - \cos\theta\right]^2 + \left[2n\lambda_o - \sin\theta\right]^2\right\}} \quad \ldots \quad (3.24)$$

$$= 4 \quad F \quad \left|\frac{p_{21}}{p_{12}}\right| \left(1 - \sqrt{\frac{n_i}{n}}\right)^2 \quad \ldots \quad (3.25)$$

where

$$F = \frac{n}{\left\{\left[n(1-\lambda_o^2) - \cos\theta\right]^2 + \left[2n\lambda_o - \sin\theta\right]^2\right\}} \quad (3.26)$$

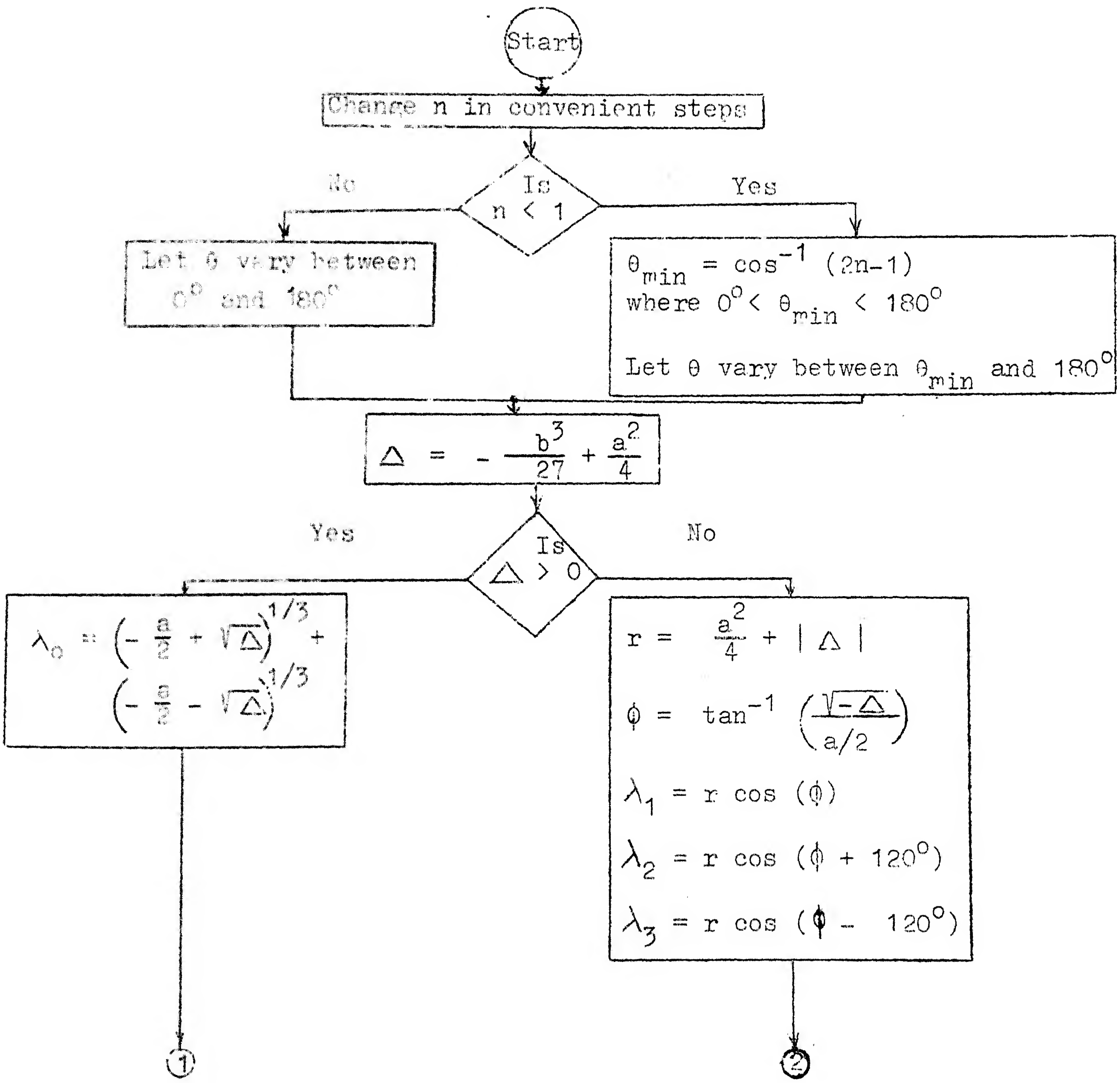
To design an amplifier for a given stability, a suitable value ($> \prod$) of the performance factor n is chosen. From the value of θ (Eqn. 2.6) and the value of n, the value of the optimum root λ_o is calculated. Once λ_o is known, the real parts ρ_S, ρ_L of the terminations and the maximum operating power gain $g_{max\ n}$ are calculated from Eqns. (3.21), (3.22), (3.25) and (3.26). To calculate the imaginary parts σ_S, σ_L of the terminations, Eqns. (3.6) are rewritten for the optimized situation.

$$\lambda_1 = \lambda_2 = \lambda_o = \frac{\sigma_{11} + \sigma_S}{\rho_{11} + \rho_S} = \frac{\sigma_{22} + \sigma_L}{\rho_{22} + \rho_L} \quad \ldots \quad (3.27)$$

If the values of λ_o and F as functions of n and θ are given in the form of graphs they can be read directly. The next section deals with the computer solution of λ_o and F vs. θ.

3.2 The Computer Solution of λ_o and F:

The flow chart for calculation of λ_o and the factor F of Eqn. (3.26) is shown in Fig. 3.1 below:



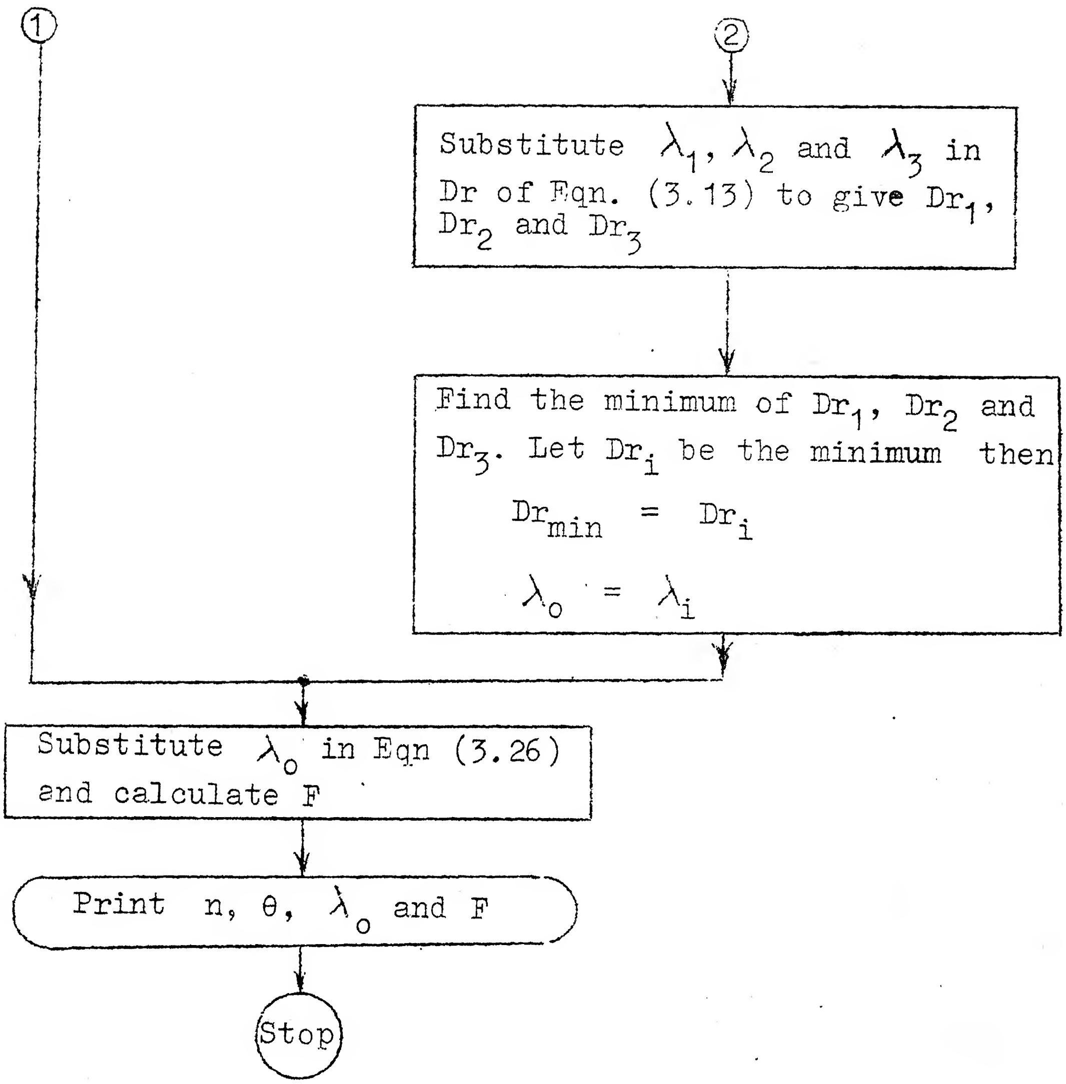


Fig. 3.1

In the flow chart above, only the values of θ between 0^o and 180^o are considered. For values of θ lying between -0^o and -180^o, the root is given by

$$\lambda_o(-\theta) = -\lambda_o(\theta) \qquad \ldots \quad (3.28)$$

from an inspection of the cubic equation (3.15). A program was written according to the flow chart, in Fortran IV and run on IBM 7044. The results showed that the cubic equation (3.15) had a single positive real root. The other two roots were either both negative or complex conjugate. In the former case (of 3 real roots) the only positive root gave the highest value for F and hence was the optimum root λ_o. From these results, plots of λ_o vs. θ were made[12,13] as shown in Fig. 3.2. In Fig. 3.3, F is plotted as a function of θ for various values of n.

In Fig. 3.2 the curves for $n > 10$ suggest[12] the approximation

$$\lambda_o \simeq \frac{\sin\theta}{n} \qquad \text{for } n \geqslant 10 \qquad \ldots(3.29)$$

A better approximation[12] which is good enough right down to $n = 5$ is

$$\lambda_o \simeq \frac{\sin\theta}{n + \cos\theta} \qquad \text{for } n \geqslant 5 \qquad \ldots(3.30)$$

This is also seen from Eqn. (3.15) which is repeated below

$$\lambda^3 - b\lambda - a = 0 \qquad \ldots(3.15)$$

where

$$b = -\left(1 + \frac{\cos\theta}{n}\right) \quad ; \quad a = \frac{\sin\theta}{n}$$

If $a = 0$ the three roots are $+\sqrt{b}$, $-\sqrt{b}$ and 0. If a is not zero but a small quantity i.e. if $a \ll 1$ as happens when $n \gg 1$ then the roots are $+\sqrt{b}$, $-\sqrt{b}$ and ϵ where $\epsilon \ll 1$. The value of ϵ can

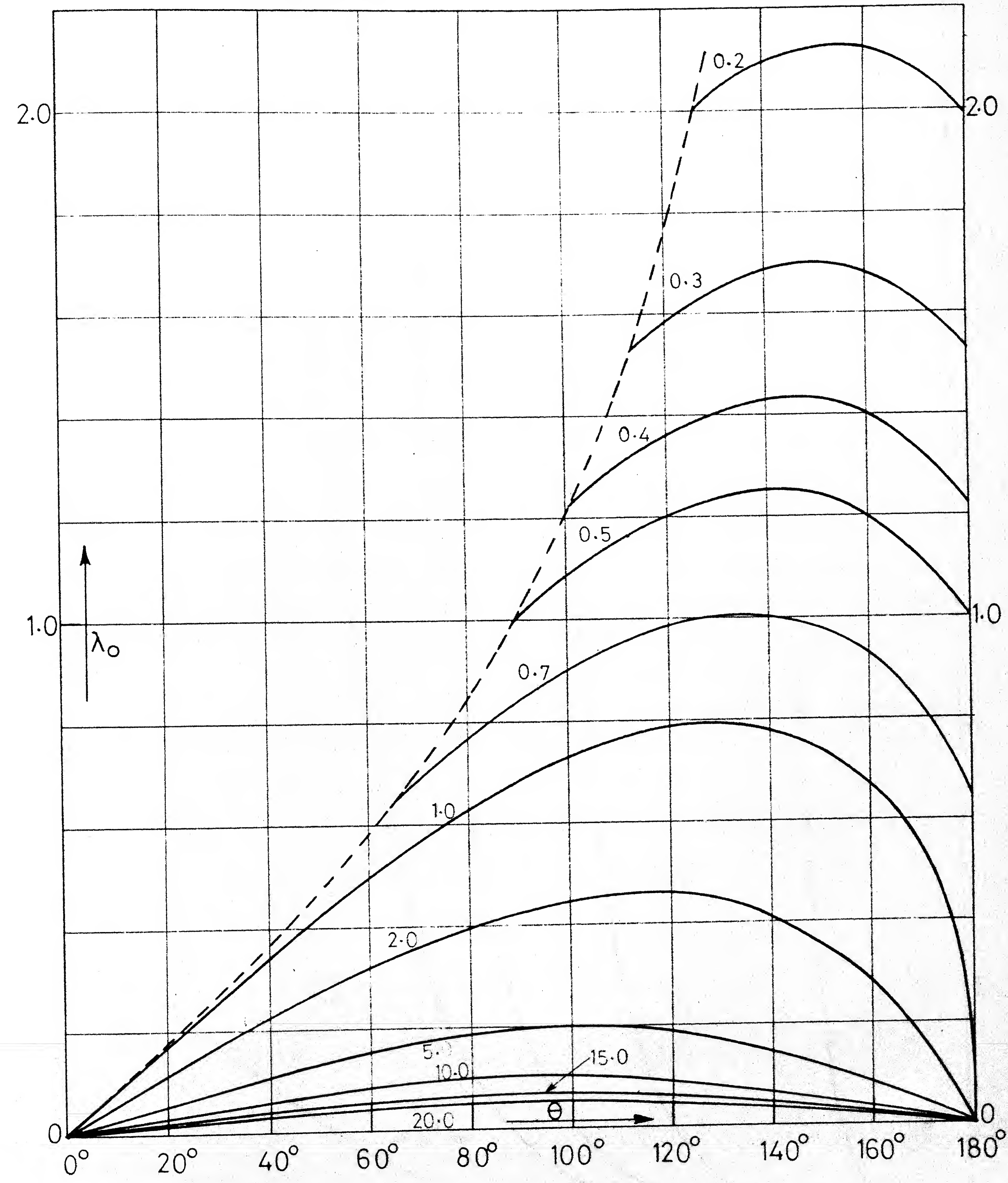


FIG. 3.2 VARIATION OF λ_o WITH θ FOR VARIOUS VALUES OF n

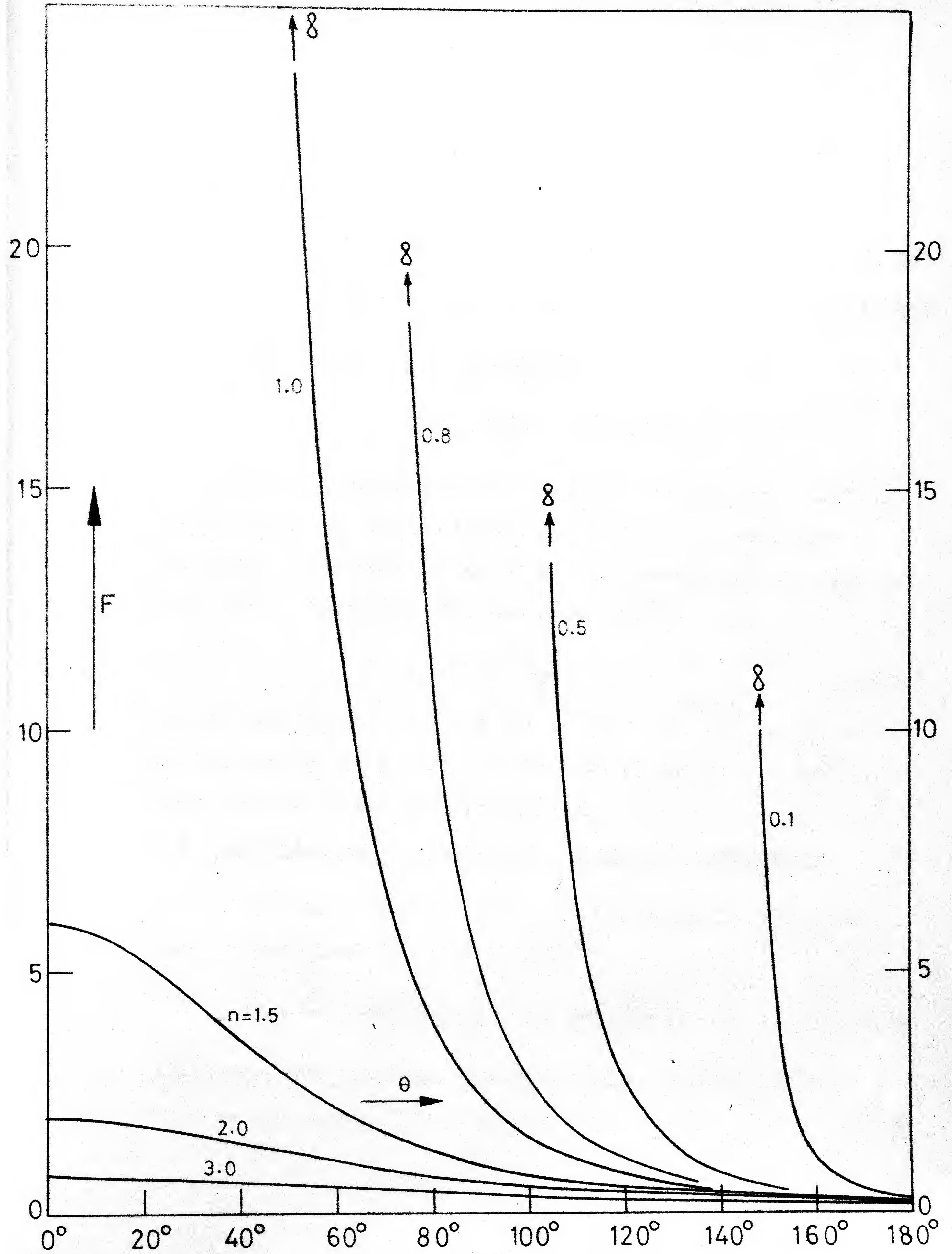


FIG. 3.3 VARIATION OF F WITH θ FOR VARIOUS VALUES OF n

be found from

$$(\lambda - \sqrt{b})(\lambda + \sqrt{b})(\lambda - \epsilon) \simeq \lambda^3 - b\lambda - a \qquad \ldots(3.31)$$

$$\therefore \quad -\sqrt{b} \times \sqrt{b} \times \epsilon \simeq a \qquad \ldots(3.32)$$

$$\therefore \epsilon \simeq -\frac{a}{b} = \frac{\sin\theta}{n + \cos\theta}$$

$$\simeq \frac{\sin\theta}{n} \quad \text{for large } n \qquad \ldots(3.33)$$

Another approximation can easily be made for small n. For $n = 0.1$, λ_o varies between 3 and 3.09 -a variation of 3% only. For lower values of n, the percentage variation is even less. Therefore, the approximation[13]

$$\lambda_o \simeq \sqrt{\frac{1-n}{n}} \qquad \ldots(3.34)$$

may be made for $n < 0.1$ and $0 < \theta < \pi$. $+\sqrt{\frac{1-n}{n}}$ is the value of the root λ_o at $\theta = \theta_{min} = \cos^{-1}(2n-1)$ and at $\theta = 180^o$ as seen from the cubic equation (3.15).

3.3 Variation of λ_o with k, η, S and δ as Parameters:

The cubic equation which results when the operating gain is maximized for a given k is[14]

$$\lambda^3 + \left[1 + \frac{2\cos\theta}{k(1+\cos\theta)}\right]\lambda - \frac{2\sin\theta}{k(1+\cos\theta)} = 0 \qquad \ldots(3.35)$$

Similarly, the following equations result when η, S or δ is used as the measure of stability.

$$\lambda^3 + \left[1 + \frac{2\cos\theta}{\eta + \cos\theta}\right]\lambda - \frac{2\sin\theta}{\eta + \cos\theta} = 0 \qquad \ldots(3.36)$$

$$\lambda^3 + \left[1 + \frac{4\ S \cos\theta}{4\ \Pi\ S + (S-1)^2}\right]\lambda - \frac{4\ S \sin\theta}{4\ \Pi\ S + (S-1)^2} = 0 \qquad \ldots(3.37)$$

and
$$\lambda^3 + \left[1 + \frac{\cos\theta}{\delta + \Pi}\right]\lambda - \frac{\sin\theta}{\delta + \Pi} = 0 \qquad \ldots(3.38)$$

A similar program is used to compute λ_o with k, η , S or δ as parameter. One change in the program is that θ is always varied from 0^o to 180^o, unlike the program for n, where variation of θ depends upon the value of n. Figs. 3.4 to 3.7 show the plots of λ_o vs. θ.

The curves for high values of η , S or δ like those of n appear to be sinusoidal and the following approximations can be made.

$$\lambda_o \simeq -\frac{a}{b} = \frac{2 \sin\theta}{\eta + 3 \cos\theta} \qquad \ldots(3.39)$$

$$\lambda_o \simeq -\frac{a}{b} = \frac{4\ S \sin\theta}{4\ \Pi\ S+(S-1)^2 + 4S \cos\theta} \qquad \ldots(3.40)$$

$$\lambda_o \simeq -\frac{a}{b} = \frac{\sin\theta}{\delta + \Pi + \cos\theta} \qquad \ldots(3.41)$$

No simple approximation for high values of k is possible. Approximations for low values of k, η , S or δ are not evident from the figures, though they can be found analytically (Appendix M) as

$$\lambda_o \simeq \frac{\tan\left(\frac{\theta}{2}\right)}{\sqrt{k}} \qquad \ldots(3.42)$$

$$\lambda_o \simeq \frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{\eta + \cos\theta}} \qquad \ldots(3.43)$$

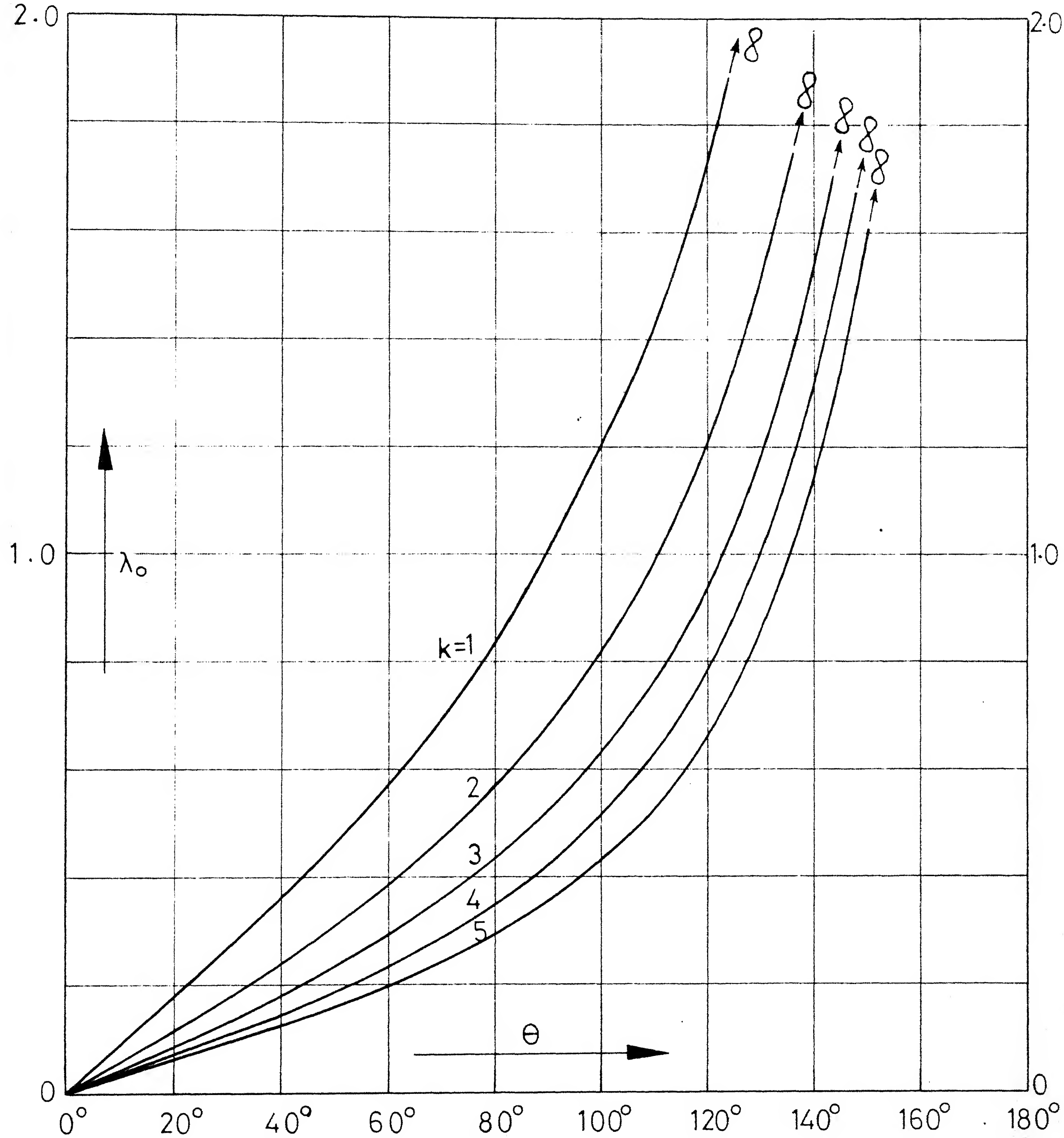


FIG. 3.4 VARIATION OF λ_o WITH θ FOR VARIOUS VALUES OF **k**

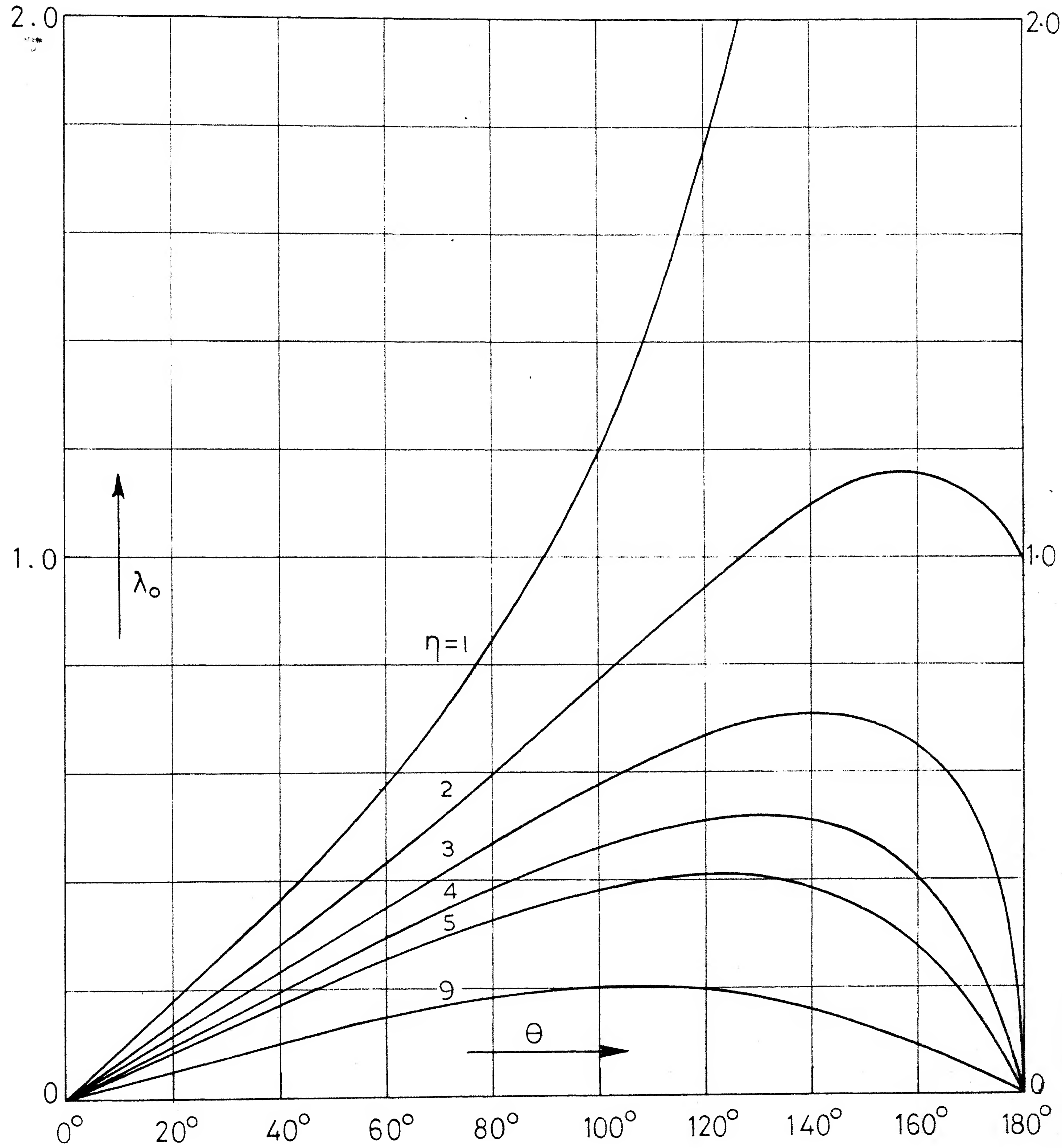


FIG. 3·5 VARIATION OF λ_o WITH θ FOR VARIOUS VALUES OF η

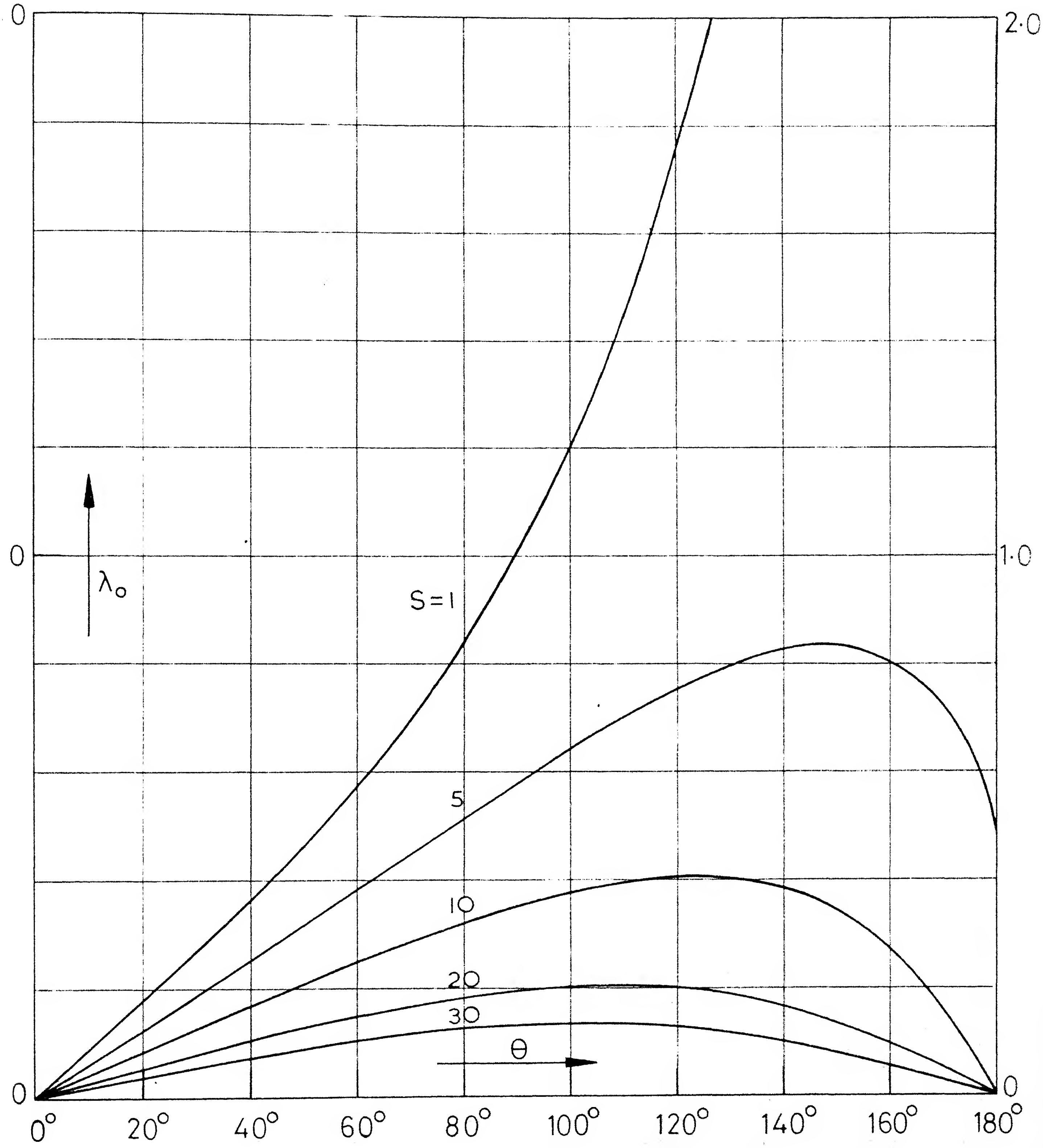


FIG. 3.6 VARIATION OF λ_o WITH θ FOR VARIOUS VALUES OF S

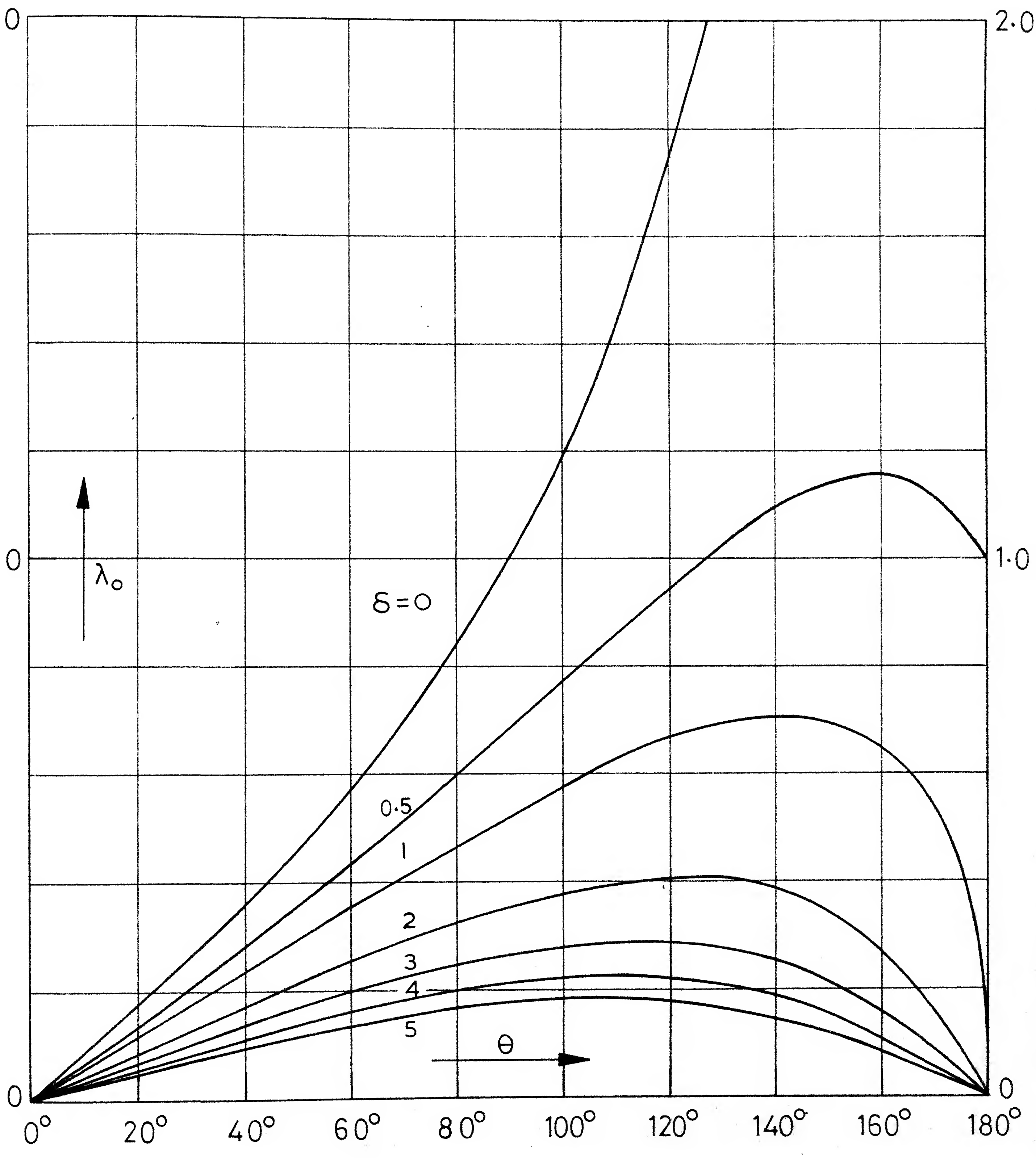


FIG. 3.7 VARIATION OF λ_o WITH θ FOR VARIOUS VALUES OF δ

$$\lambda_o \simeq \frac{2\sqrt{S}\,\sin\left(\frac{\theta}{2}\right)}{\sqrt{4\Pi S + (S-1)^2}} \qquad \ldots(3.44)$$

$$\lambda_o \simeq \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{\delta + \Pi}} \qquad \ldots(3.45)$$

For $k = 1 = \eta = S$, $\delta = 0$ all the 4 expressions above reduce to

$$\lambda_o = \tan\left(\frac{\theta}{2}\right) \qquad \ldots(3.46)$$

CHAPTER 4

ANALYTICAL SOLUTION OF λ_o

In this chapter, the dependence of λ_o, the optimum real root of Eqn. (3.15) on θ and n will be discussed. The approximate relations (3.29) to (3.31) will be derived analytically. The errors involved in the approximations will also be calculated.

Before proceeding with the analytical solution, it will be helpful to know the conditions under which the cubic equation has 3 real roots and the signs and magnitudes of these roots.

4.1 Number of Real Roots, Their Signs and Magnitudes:

The cubic equation (3.15) is repeated below for convenience.

$$\lambda^3 - b\lambda - a = 0 \qquad \ldots \quad (4.1)$$

where

$$b = -\left(1 + \frac{\cos\theta}{n}\right) \quad ; \quad a = \frac{\sin\theta}{n} \qquad \ldots \quad (4.2)$$

It has been shown[15,16] that the cubic equation has a single real root for $n \geqslant 1$. The number of real roots of a cubic equation like (4.1) depends upon the Δ. The Δ is defined as[17]

$$\Delta = -\frac{b^3}{27} + \frac{a^2}{4} \qquad \ldots \quad (4.3)$$

There is a single real root for $\triangle > 0$ and 3 real roots for $\triangle \leqslant 0$. Thus, inspection of Eqns. (4.3) and (4.2) shows that there is a single real root for $n > 1$. For $n \leqslant 1$ there may be a single real root or 3 real roots depending upon θ.

Let the 3 roots of the cubic equation be denoted by λ_1, λ_2 and λ_3. Then,

$$\lambda_1 \lambda_2 \lambda_3 = a \quad \ldots \quad (4.4)$$

$$\lambda_1 + \lambda_2 + \lambda_3 = 0 \quad \ldots \quad (4.5)$$

and

$$\lambda_1 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_1 = -b \quad \ldots \quad (4.6)$$

Suppose that all the roots are real. Then Eqn.(4.5) shows that all the roots cannot have the same sign. There are two possibilities - one positive and two negative roots or one negative and two positive roots. Then, Eqn. (4.4) shows that the first combination of signs is allowed when $a > 0$ and the second when $a < 0$. Let the case of $a > 0$ be considered where λ_1 denotes the positive root and λ_2, λ_3 the negative roots. Now Eqn.(4.6) may be modified as

$$\lambda_1(\lambda_2 + \lambda_3) + \lambda_2 \lambda_3 = -b \quad \ldots \quad (4.7)$$

Therefore,

$$-\lambda_1^2 + \lambda_2 \lambda_3 = -b \quad \ldots \quad (4.8)$$

using Eqn. (4.5)

so that,

$$\lambda_1^2 = b + \lambda_2 \lambda_3 \quad \ldots \quad (4.9)$$

Similarly,

$$\lambda_2^2 = b + \lambda_3 \lambda_1 \quad \ldots \quad (4.10)$$

$$\lambda_3^2 = b + \lambda_1 \lambda_2 \quad \ldots \quad (4.11)$$

Since λ_1 is positive and λ_2, λ_3 negative, therefore $\lambda_2\lambda_3$ of Eqn. (4.9) is positive and $\lambda_3\lambda_1$, $\lambda_1\lambda_2$ of Eqns. (4.10) and (4.11) are negative. Then inspection of Eqns. (4.10), and (4.11) shows that b must be positive and have sufficiently high value so that the r.h.s. of both the equations are positive and yield real values for λ_2 and λ_3. In other words, sufficiently positive value of b makes Δ of Eqn. (4.3) negative, giving 3 real roots. A positive b implies that,

$$\lambda_1^2 > b, 0 < \lambda_2^2 < b \ \& \ 0 < \lambda_3^2 < b \quad \ldots \quad (4.12)$$

These inequalities give,

$$\lambda_1 > +\sqrt{b}, \quad -\sqrt{b} < \lambda_2 < 0 \text{ and } -\sqrt{b} < \lambda_3 < 0 \quad (4.13)$$

Suppose a < 0. Then there is one negative root and two positive roots as shown earlier. If λ_1 denotes the negative root and λ_2, λ_3 the positive roots, the inequalities (4.12) once again follow from Eqns. (4.9) to (4.11). In this case,

$$\lambda_1 < -\sqrt{b}, \quad 0 < \lambda_2 < \sqrt{b} \text{ and } 0 < \lambda_3 < \sqrt{b} \quad (4.14)$$

Consider the case of a single real root, λ_1, say, the other two roots being complex conjugate. Then Eqn. (4.4) shows that the only real root has the same sign as a, as the product of two complex conjugate numbers is a positive quantity. Also Eqn (4.9) gives $\lambda_1^2 > b$ so that, $\lambda_1 > +\sqrt{b}$ or $\lambda_1 < -\sqrt{b}$. provided b > 0. The first inequality holds for a > 0 and the second for a < 0.

The above results may also be visualized graphically.[13] Let the cubic equation (4.1) be rearranged as,

$$\lambda = \frac{a}{\lambda^2 - b} \qquad \ldots \quad (4.15)$$

The roots of the cubic equation (4.1) therefore, correspond to the points of intersection of the two curves

$$y = \lambda \quad \text{and} \quad y = f(\lambda) = \frac{a}{\lambda^2 - b} \qquad \ldots \quad (4.16)$$

The curves are shown in Figs. (4.1). Fig. 4.1(a) shows that there is a single point of intersection i.e. a single positive real root λ_1 for b < 0. Fig 4.1(b) shows that there is still a single positive real root λ_1, such that $\lambda_1 > +\sqrt{b}$. Fig.4.1(c) shows that there is a single positive root λ_1 and two negative roots λ_2 and λ_3 when b is sufficiently positive. The roots are such that $\lambda_1 > +\sqrt{b}$ and $0 > \lambda_2, \lambda_3 > -\sqrt{b}$. Figs. 4.2 are drawn for a < 0 and similar conclusions follow from these figures too.

Now, it is shown in Appendix A that, for a > 0, the single positive root gives the maximum value for F of Eqn. (3.26). Similarly, for a < 0, the single negative root gives the maximum value for F. In the sections that follow, approximate expressions for the root of the cubic Eqn. (4.1) are derived. Since they yield positive values for a > 0 and negative values for a < 0, they represent the approximate expressions for the optimum root λ_0.

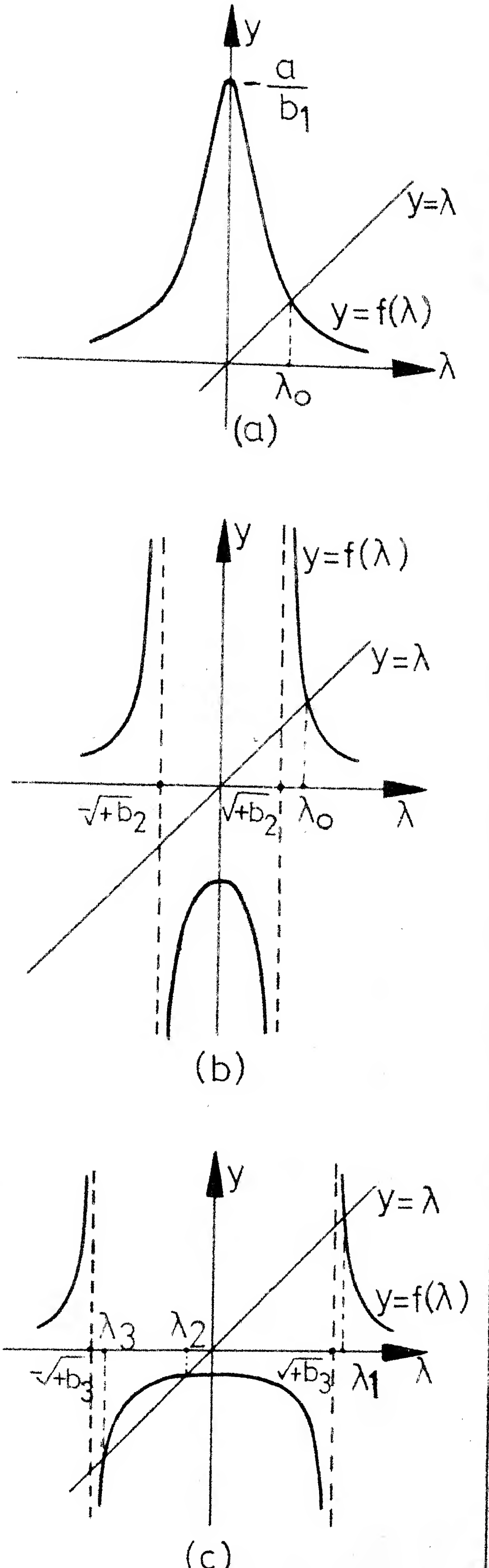


FIG. 4·1 ($a > o$) NUMBER OF REAL ROOTS FOR VARIOUS VALUES OF b SUCH THAT $b_1 < o < b_2 < b_3$

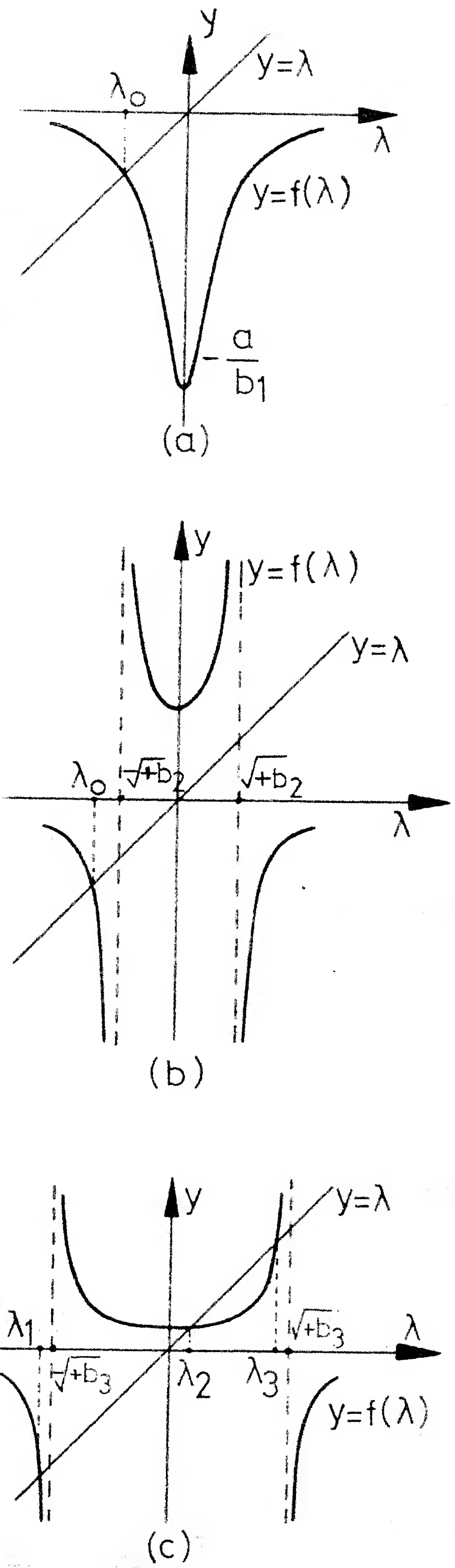


FIG. 4·2 ($a < o$) NUMBER OF REAL ROOTS FOR VARIOUS VALUES OF b SUCH THAT $b_1 < o < b_2 < b_3$

4.2 Approximate Expression For $n \geq 5$

Let the cubic Eqn. (4.1) be rewritten as,

$$\lambda^3 + c\lambda - a = 0 \qquad \ldots \quad (4.17)$$

where,

$$c = -b = 1 + \frac{\cos\theta}{n} \; ; \quad a = \frac{\sin\theta}{n} \qquad \ldots \quad (4.18)$$

The cubic equation may be rearranged as,

$$\lambda = \frac{a}{c} - \frac{\lambda^3}{c} = \frac{a}{c}\left(1 - \frac{\lambda^3}{a}\right) \qquad \ldots \quad (4.19)$$

or as,

$$= \frac{a}{c + \lambda^2} \qquad \ldots \quad (4.20)$$

These are iterative formulas for a root, real or complex of the cubic equation (4.17). Equation (4.19) may be rewritten as,

$$\lambda = \frac{a}{c}(1 - X_1) \qquad \ldots \quad (4.21)$$

where,

$$X_1 = \frac{\lambda^3}{a} \qquad \ldots \quad (4.22)$$

If the value $\lambda = \frac{a}{c} - \frac{\lambda^3}{c}$ is substituted in the right hand side of Eqn (4.19) then,

$$\lambda = \frac{a}{c} - \frac{1}{c}\left(\frac{a}{c} - \frac{\lambda^3}{c}\right)^3 \qquad \ldots \quad (4.23)$$

$$= \frac{a}{c}(1 - T + X_2) \qquad \ldots \quad (4.24)$$

where,

$$X_2 = \frac{a\lambda^3}{c^3}\left(3 - \frac{3\lambda^3}{a} + \frac{\lambda^6}{a^2}\right) \qquad \ldots \quad (4.25)$$

and

$$T = \frac{a^2}{c^3} = \frac{n\sin^2\theta}{(n+\cos\theta)^3} \qquad \ldots \quad (4.26)$$

If successive substitutions as in Eqn. (4.23) are made, the equations as shown below result. For convenience, Eqns.(4.21) and (4.24) are repeated.

$$\left.\begin{aligned}
\lambda &= \frac{a}{c}(1 - X_1) && a\\
\lambda &= \frac{a}{c}(1 - T + X_2) && b\\
\lambda &= \frac{a}{c}(1 - T + 3T^2 - X_3) && c\\
\lambda &= \frac{a}{c}(1 - T + 3T^2 - 12T^3 + X_4) && d\\
\lambda &= \frac{a}{c}(1 - T + 3T^2 - 12T^3 + 55T^4 - X_5) && e
\end{aligned}\right] \quad ..(4.27)$$

In Appendix B it is shown that if λ represents a real root and if $T \ll 1$ which occurs for high values of n (as shown in Appendix C) then X_1 and X_2 are sufficiently small and may be neglected. Similarly it can be shown that X_3, X_4, X_5 etc. are small for $T \ll 1$ and may be neglected. Then, the following approximate expressions result. As there is only one real root for $n > 1$, it is the same as the optimum root λ_o

$$\left.\begin{aligned}
\lambda_o &\simeq \frac{a}{c} && a\\
\lambda_o &\simeq \frac{a}{c}(1 - T) && b\\
\lambda_o &\simeq \frac{a}{c}(1 - T + 3T^2) && c\\
\lambda_o &\simeq \frac{a}{c}(1 - T + 3T^2 - 12T^3) && d\\
\lambda_o &\simeq \frac{a}{c}(1 - T + 3T^2 - 12T^3 + 55T^4) && e
\end{aligned}\right] \quad ..(4.28)$$

where $$T = \frac{n \sin^2 \theta}{(n + \cos \theta)^3}$$

Denoting the exact value of the optimum root by λ_{oe}, the error in the approximate value λ_o is given by,

$$\mathcal{E} = \frac{\lambda_o - \lambda_{oe}}{\lambda_{oe}} \times 100 \quad \% \qquad \ldots \quad (4.29)$$

Noting that λ_{oe} is given by any of the expressions (4.27), the errors (in per unit) in the approximate expressions (4.28) are respectively,

$$\left.\begin{aligned}
\mathcal{E}_\lambda &= \frac{X_1}{1 - X_1} \simeq X_1 \simeq T && a\\
\mathcal{E}_\lambda &= \frac{X_2}{1 - X_1} \simeq X_2 \simeq 3T^2 && b\\
\mathcal{E}_\lambda &= \frac{X_3}{1 - X_1} \simeq X_3 \simeq 12T^3 && c\\
\mathcal{E}_\lambda &= \frac{X_4}{1 - X_1} \simeq X_4 \simeq 55T^4 && d\\
\mathcal{E}_\lambda &= \frac{X_5}{1 - X_1} \simeq X_5 \simeq 273T^5 && e
\end{aligned}\right] \ldots (4.30)$$

The above expressions are valid for small T which also implies small X_1 (Appendix B). The errors (4.30) show that (4.28) are a set of better and better approximations. For example, if T, a function of n and θ is restricted to values < 0.05 by keeping n > 4.7 (Appendix C) then the errors (4.30) are < 5, 0.75, 0.15, 0.035 and 0.0085 per cent respectively. If T < 0.1 corresponding to the range n > 3.48 then the errors are < 10, 3, 1.2, 0.55 and 0.273 per cent respectively. This shows that with a higher value of T not only are the errors higher but they decrease less rapidly. For even higher values of T the approximations (4.28)

give larger and larger errors. Though this cannot be proved directly as it is difficult to estimate X_1, X_2, X_3 etc. for large values of T, an inspection of one of the expressions (4.28)d say will prove the point.

$$\lambda_o = \frac{a}{c}(1 - T + 3T^2 - 12T^3)$$

For T = 0.5, the magnitudes of the terms in the parenthesis are 1, 0.5, 0.75, 1.5 respectively. This is a diverging sequence. This suggests that $|X_1|$, $|X_2|$, $|X_3|$ etc. is also a diverging sequence implying that the errors increase with the successive expressions (4.28). Computer calculations for n = 1.95 giving T = 0.5 indeed show that the errors as defined by Eqn. (4.29) are 27, -28, 45, -84 and 175% respectively. Thus it is clear that the expressions (4.28) are useful for small values of T i.e. large n.

Now suppose an error of 5 % is to be allowed in an approximate expression for λ_o. Then the range of n over which expression (4.28)a is useful may be determined. From Eqn. (4.30)a it is seen that T should be $\leqslant 0.05$. Then Fig. C-1 in Appendix C shows that n must be $\geqslant 4.7$. Conversely, if $n \geqslant 5$, the error is < 4.4 %. Thus the expression

$$\lambda_o \simeq \frac{a}{c} = \frac{\sin\theta}{n + \cos\theta} \qquad \ldots \qquad (4.31)$$

may be used for $n \geqslant 5$ with an error < 4.4 %. Computer calculations show that the error is $\leqslant 4.1$ %. If the approximate expression

$$\lambda_o \simeq \frac{a}{c}(1 - T) = \frac{\sin\theta}{n + \cos\theta}\left[1 - \frac{n \sin^2}{(n + \cos\theta)^3}\right] \quad \ldots \quad (4.32)$$

is used over the same range of n viz. $n \geqslant 5$, then the error is $3T^2 \times 100$ which is < 0.58 %. Computer calculations show that the error in (4.32) is <0.51 %. Conversely, the approximate expression (4.32) may be used for n as low as 3.15 with an error < 5 %. But, for a convenient division of the range of values of n viz. $n \geqslant 5$, $0.1 < n < 5$ and $n \leqslant 0.1$, the expressions (4.31) and (4.32) may be reserved for $n \geqslant 5$.

4.3 Approximate Expression for λ_o For $n \leqslant 0.1$

The cubic equation (4.17) may be rearranged as,

$$\lambda = \pm\sqrt{\frac{a}{\lambda} - c} \quad \ldots \quad (4.33)$$

This is another iterative formula for the root, real or complex, of the cubic equation (4.17). If $a = \frac{\sin\theta}{n}$ is positive then the desired root λ_o is positive as shown earlier. In the following discussion, a is restricted to positive values. Therefore the + sign must be chosen. Then

$$\lambda = +\sqrt{\frac{a}{\lambda} - c} = \sqrt{\frac{\sin\theta}{n\lambda} - \left(1 + \frac{\cos\theta}{n}\right)} \quad \ldots \quad (4.34)$$

$$= \sqrt{\frac{1-n}{n}}\sqrt{1 + \frac{\sin\theta}{(1-n)\lambda} - \frac{1 + \cos\theta}{1-n}} \quad \ldots \quad (4.35)$$

$$= R(1 + A)^{1/2} \quad \ldots \quad (4.36)$$

where, $R = \sqrt{\frac{1-n}{n}}$ $\quad \ldots \quad (4.37)$

and $A = \frac{1}{(1-n)}\left[\frac{\sin\theta}{\lambda} - (1 + \cos\theta)\right]$ $\quad \ldots \quad (4.38)$

If the value of λ is substituted from Eqn. (4.36) into Eqn. (4.38), then

$$A = \frac{1}{(1-n)}\left[\frac{\sin\theta}{R(1+A)^{1/2}} - (1+\cos\theta)\right] \quad \ldots \quad (4.39)$$

$$= \frac{P}{(1+A)^{1/2}} - Q \quad \ldots \quad (4.40)$$

where

$$P = \frac{\sin\theta}{(1-n)R} = \frac{\sqrt{n}\sin\theta}{(1-n)^{3/2}} \;;$$

$$Q = \frac{1+\cos\theta}{1-n} \quad \ldots \quad (4.41)$$

In Eqn. (4.27) λ_o was expressed in terms of a power series in T followed by the remainder X which could be neglected. It was also shown that such an expansion in terms of T was useful for low values of T i.e. high values of n. Similarly, using repeated substitutions, λ_o may be expressed as,

$$\lambda_o = R\left[1 + \left(\frac{P-Q}{2}\right) - \left(\frac{3P^2 - 4PQ + Q^2}{8}\right) + \ldots\ldots + W\right] \quad (4.42$$

where W is a function of λ. Such an expression is useful for small values of P and Q which are attained for small n*. A number of

---

* $P_{max} = Q_{max} = \frac{2n}{1-n}$ for $n < 0.5$. This is seen as follows. For $n<1$, θ can take values between $\theta_{min} = \cos^{-1}(2n-1)$ and 180^o (or between $-\theta_{min}$ and -180^o) so as to ensure absolute stability. For $n < 0.5$, $\theta_{min} > 90^o$. Under this condition, the magnitudes of $P = \frac{\sqrt{n}\sin\theta}{(1-n)^{3/2}}$ and $Q = \frac{1+\cos\theta}{1-n}$ decrease monotonically from $\frac{2n}{1-n}$ at $\theta = \theta_{min}$ to 0 at $\theta = \pm 180^o$.

successive approximations can be deduced from Eqn. (4.33) viz.,

$$\left.\begin{aligned}\lambda_o &= R && a\\ \lambda_o &= R\left[1+\left(\frac{P-Q}{2}\right)\right] && b\\ \lambda_o &= R\left[1+\left(\frac{P-Q}{2}\right)-\left(\frac{3P^2-4PQ+Q^2}{8}\right)\right] && c\end{aligned}\right\} \quad (4.43)$$

Denoting the exact value of the optimum root by λ_{oe}, which is given by (4.36), the errors in the expressions (4.43) are given by

$$\left.\begin{aligned}\mathcal{E}_\lambda &= \frac{1}{\sqrt{1+A}} - 1 && a\\ \mathcal{E}_\lambda &= \frac{1+\left(\frac{P-Q}{2}\right)}{\sqrt{1+A}} - 1 && b\\ \mathcal{E}_\lambda &= \frac{1+\left(\frac{P-Q}{2}\right)-\left(\frac{3P^2-4PQ+Q^2}{8}\right)}{\sqrt{1+A}} - 1 && c\end{aligned}\right\} \quad (4.44)$$

Let an error of 5 % be allowed as before in the approximate expression for λ_o. The range of n over which the expression (4.43)a may be used can be determined. This is done in Appendix D in which it is shown that n must be $\leqslant 0.16$. For convenience, n may be restricted to values $\leqslant 0.1$. Then the error $\mathcal{E}_\lambda$ is < 2.95 %. Computer calculations show that the error is $\leqslant$ 2.8 % . If expression (4.43)b is used for n $\leqslant$ 0.1, then the error $\mathcal{E}_\lambda$ is < 0.366 % . as shown in Appendix E. This too is verified by computer calculations which show that the error is < 0.33 % . Thus the approximate expressions

approximate value for λ_o as $\lambda_{o2} = f(\lambda_{o1})$. This is substituted in $f(\lambda)$ again to give a third approximation $\lambda_{o3}=f(\lambda_{o2})$ and so on. If the successive values λ_{o1}, λ_{o2} etc. converge to a limit, that limit is a root of the cubic equation (4.17). If λ_{o1} is chosen properly and the right iteration function is selected from among the 4 functions, then the root thus found will be the desired root λ_o. For given a and c one iteration function may be more suitable than the other. With the wrong choice of iteration function, one may get the wrong root, real or complex, of the cubic equation. The 'asymptotic error constant' and 'order of convergence'[19] as defined below will help in deciding the suitability of an iteration function.

4.4.1 Choice of Iteration Function

Let the error ϵ_i in the i^{th} iterate be defined by

$$\epsilon_i = \lambda_{oi} - \lambda_{oe} \qquad \ldots (4.51)$$

where λ_{oe} is the exact value of the root. It is assumed that the successive iterates converge to λ_{oe} so that

$$\lim_{i \to \infty} \lambda_{oi} = \lambda_{oe} \quad \text{and} \quad \lim_{i \to \infty} \epsilon_i = 0 \qquad \ldots (4.52)$$

If there exists a number p such that

$$\lim_{i \to \infty} \frac{|\epsilon_{i+1}|}{|\epsilon_i|^p} = C \qquad \ldots (4.53)$$

then the iteration function is said to have an order of convergence p at λ_{oe} and the limit C is called the asymptotic error constant (a.e.c.) When p = 1, the iteration function

is said to have a linear convergence. p and C depend upon the derivatives of $f(\lambda)$ at λ_{oe} as shown below

$$\text{If } f'(\lambda_{oe}) \neq 0 \qquad \text{then } p = 1, C = |f'(\lambda_{oe})|$$

$$\text{If } f'(\lambda_{oe}) = 0;\ f''(\lambda_{oe}) \neq 0 \text{ then } p = 2, C = \frac{1}{\lfloor 2}|f''(\lambda_{oe})|$$

$$\text{If } f'(\lambda_{oe}) = 0 = f''(\lambda_{oe});$$

$$f'''(\lambda_{oe}) \neq 0 \qquad \text{then } p = 3, C = \frac{1}{\lfloor 3}|f'''(\lambda_{oe})|$$

etc.... (4.54)

When $p = 1$, C must be < 1 in order that there be convergence. It can be shown that all the 4 functions (4.47) to (4.50) are linearly convergent and that the respective a.e.cs are

$$C_1 = \frac{1}{C_2} = \left|\frac{2\lambda_{oe}^3}{a}\right| \qquad \ldots (4.35)$$

and

$$C_3 = \frac{1}{C_4} = \left|\frac{3\lambda_{oe}^2}{c}\right| \qquad \ldots (4.56)$$

As λ_{oe} is a function of n and θ, so also C_1, C_2, C_3 and C_4 are function of n and θ. It is shown in Appendix F that

$$\frac{\left(1 - \frac{C_1}{2}\right)^3}{\frac{C_1}{2}} = \frac{\left(1 - \frac{1}{2C_2}\right)^3}{\frac{1}{2C_2}} = \frac{1}{T} \qquad \ldots (4.5)$$

$$\frac{1}{\pm\frac{C_3}{3}\left(\frac{C_3}{3} \pm 1\right)^2} = \frac{1}{\pm\frac{1}{3C_4}\left(\frac{1}{3C_4} \pm 1\right)^2} = \frac{1}{T} \qquad (4.58)$$

where

$$\frac{1}{T} = \frac{c^3}{a^2} = \frac{(n+\cos\theta)^3}{n\sin^2\theta}$$

Fig. 4.3 shows curves of constant $-\frac{1}{T}$ in the n-θ plane. It is clear from the figure that in the range of n under consideration viz. 0.1 to 5, $\frac{1}{T}$ varies from $-\infty$ to $+\infty$. A plot of an asymptotic constant C, against $\frac{1}{T}$ will give an idea of the variation of C over the range of n. Fig. 4.4 shows the variations of the a.e.c.s with $\frac{1}{T}$. To get a complete picture of the variations, the range 1 to ∞ (and -1 to $-\infty$) of $\frac{1}{T}$ is compressed in a length of one unit. This is done by actually using the variable T instead of $\frac{1}{T}$ on the x-axis as shown in the figure. A similar compression of range is done on the y-axis also. The figure shows that none of a.e.c.s remain below unity over the entire range of $\frac{1}{T}$ from $-\infty$ to $+\infty$. This means that none of the iteration functions, taken individually, ensures converging iterates over the range $0.1 < n < 5$. But Figs. 4.5 suggest that two iterations functions may be used together to ensure convergence. For example, Fig. 4.5(a) shows a scheme in which iteration function $f_2(\lambda)$ is used for $-\infty < \frac{1}{T} < 0.25$ and $f_1(\lambda)$ for $0.25 < \frac{1}{T} < \infty$. Thus convergence is guaranteed over the entire range of $\frac{1}{T}$. There are two more schemes as shown in Figs. 4.5(b) and (c) which ensure convergence for all $\frac{1}{T}$. One disadvantage common to all the 3 schemes is that the expression $\frac{1}{T} = \frac{(n+\cos\theta)^3}{n\sin^2\theta}$ must be evaluated in order to select the appropriate iteration

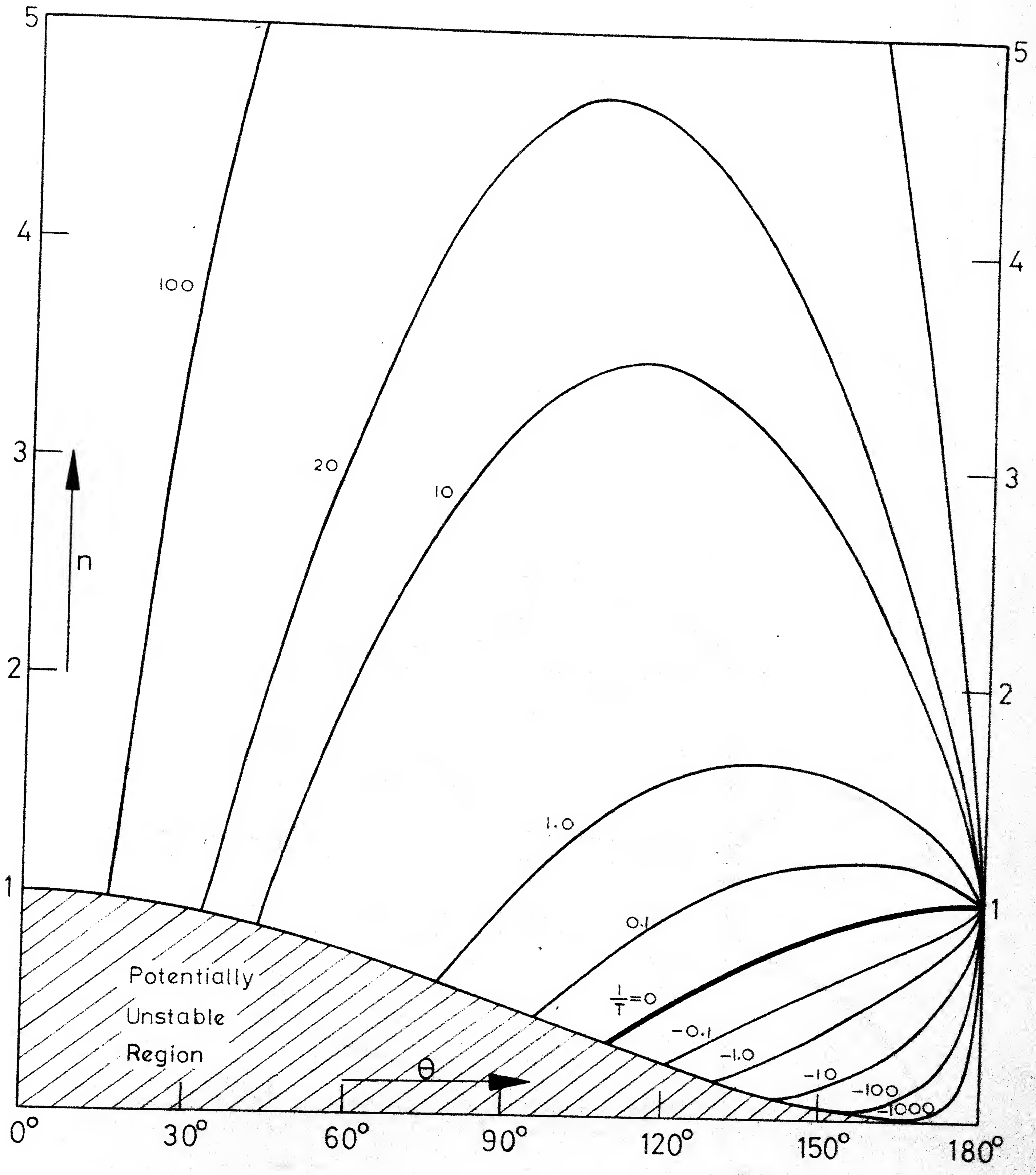


FIG. 4.3 CURVES OF CONSTANT-T IN n - θ PLANE

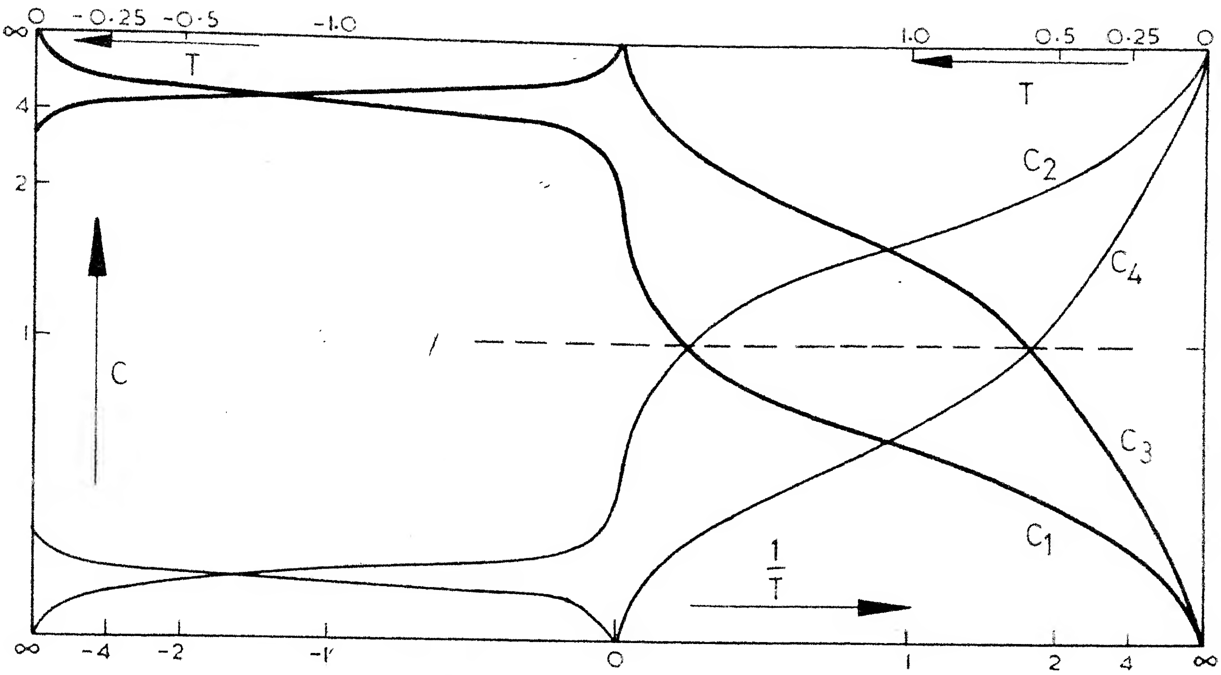


FIG. 4.4 VARIATIONS OF THE ASYMPTOTIC ERROR CONSTANTS WITH $\frac{1}{T}$

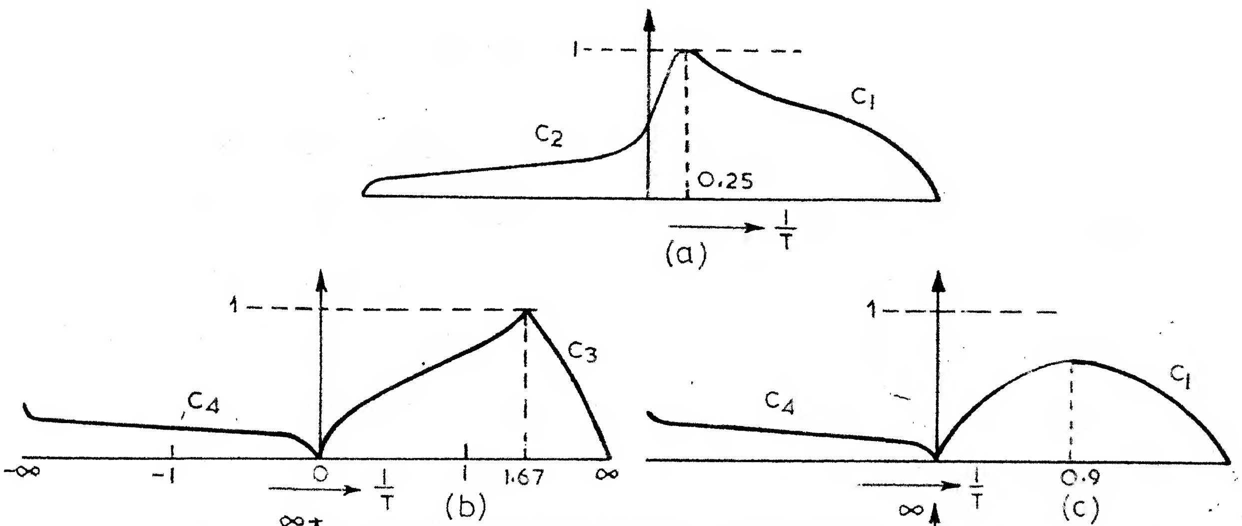


FIG. 4.5 SCHEME ENSURING CONVERGENCE OVER THE ENTIRE RANGE OF $\frac{1}{T}$

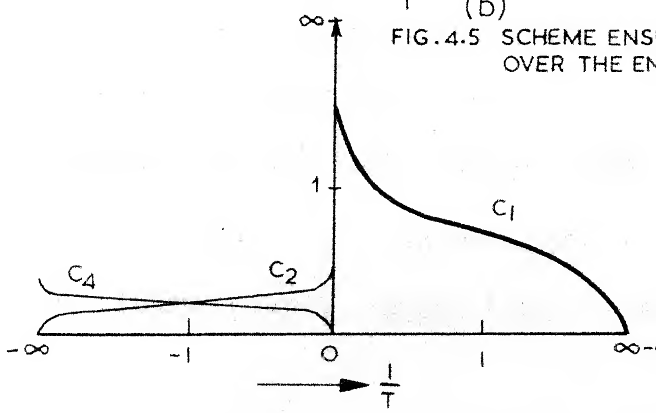


FIG. 4.6 SCHEME USING $f_1(\lambda)$ AND EITHER $f_2(\lambda)$ OR $f_4(\lambda)$

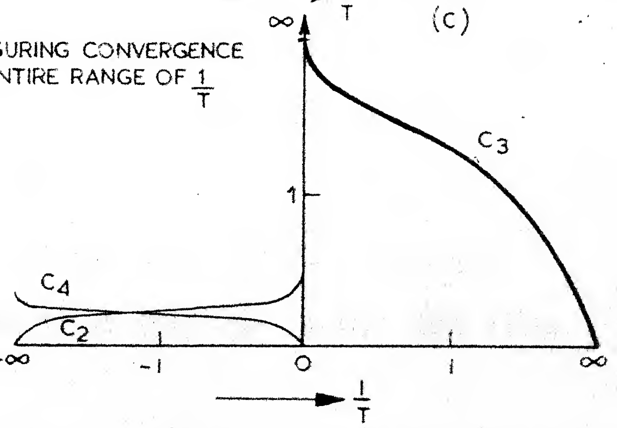


FIG. 4.7 SCHEME USING $f_3(\lambda)$ AND EITHER $f_2(\lambda)$ OR $f_4(\lambda)$

function. Figure 4.6 shows a simpler scheme in which the range of $\frac{1}{T}$ is divided about the point $\frac{1}{T} = 0$ i.e. $n + \cos\theta = 0$. The function $f_2(\lambda)$ or $f_4(\lambda)$ is used for $n + \cos\theta < 0$ and $f_1(\lambda)$ for $n + \cos\theta > 0$. Thus it is enough to calculate the expression $n + \cos\theta$ in order to select the appropriate iteration function. Admittedly, use of $f_1(\lambda)$ gives diverging iterates in the range $0 < \frac{1}{T} < 0.25$. But this can be overcome by means of the following techniques.

Firstly, the value of the first approximation λ_{01}, is chosen as

$$\lambda_{01} = (a)^{\frac{1}{3}} = \left(\frac{\sin\theta}{n}\right)^{\frac{1}{3}} \qquad (4.59)$$

This is the exact value of the optimum root for $\frac{1}{T} = 0$ i.e. $n + \cos\theta = 0$ as seen from Eqn. (4.17). As the exact value of the root in the range $0 < \frac{1}{T} < 0.25$ is close to that given by above equation, the divergence is small for the first few iterations. Secondly, 'Aitken's δ^2 process'[19] is used for accelerating the convergence. This not only takes care of the divergence but also reduces the number of calculations in the region away from $\frac{1}{T} = 0$, where λ_{01} as given by Eqn. (4.59) is not a good approximation.

Fig. 4.7 shows another scheme with $\frac{1}{T} = 0$ as the dividing point. Here, $f_3(\lambda)$ is used for $\frac{1}{T} > 0$. But this

scheme is not feasible as the divergence within the range $0 < \frac{1}{T} < 1.67$ is too rapid and the δ^2 process is of no help.

Thus the scheme of Fig. 4.6 is the best. Though $f_2(\lambda)$ and $f_4(\lambda)$ seem equally suitable in this scheme, $f_2(\lambda)$ is preferable as it is more convenient to calculate a square root than a cube root.

4.4.2 Aitken's δ^2 Process

This process of accelerating the convergence can be understood with the help of Fig. 4.8 in which $y = f_1(\lambda)$ is plotted for the case of $c = 1 + \frac{\cos\theta}{n} > 0$. Choosing a value λ_{01} corresponds to selecting a point $P_1(\lambda_{01}, \lambda_{01})$ on the line $y = \lambda$. The substitution $f(\lambda_{01}) = \lambda_{02}$ corresponds to locating the point $Q_1(\lambda_{01}, \lambda_{02})$. Similarly, $f(\lambda_{02}) = \lambda_{03}$ corresponds to locating the point $Q_2(\lambda_{02}, \lambda_{03})$. A further substitution would give the point Q_3. Instead, if one 'predicts' a point P_1' one obtains the point Q_1' which is closer to X than Q_3 is. Thus the points Q_1, Q_2, Q_1', Q_2' etc. converge faster. P_1' is the point of intersection of the lines $Q_1 Q_2$ and $y = \lambda$. From the similar triangles $Q_2 P_2 Q_1$ and $P_1' B Q_1$

$$\frac{Q_2P_2}{P_2Q_1} = \frac{P_1'B}{BQ_1} \qquad \ldots (4.60)$$

$Q_1 \equiv (\lambda_{o1}, \lambda_{o2})$ $\quad P_1 \equiv (\lambda_{o1}, \lambda_{o1})$

$Q_2 \equiv (\lambda_{o2}, \lambda_{o3})$ $\quad P_2 \equiv (\lambda_{o2}, \lambda_{o2})$

$Q_3 \equiv (\lambda_{o3}, \lambda_{o4})$ $\quad P_3 \equiv (\lambda_{o3}, \lambda_{o3})$

$P_1' \equiv ({}^1\lambda_{o1}, {}^1\lambda_{o1})$

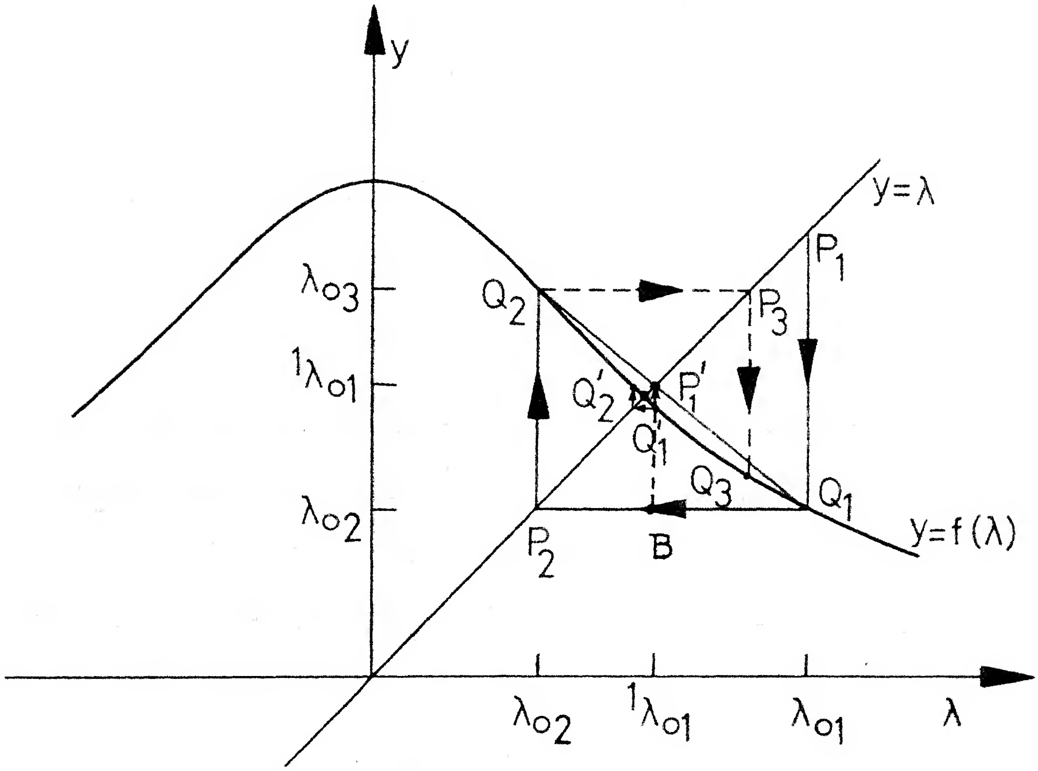


FIG.4.8 SOLUTION OF THE ROOT BY ITERATION

$$\frac{\lambda_{o3} - \lambda_{o2}}{\lambda_{o1} - \lambda_{o2}} = \frac{{}^{1}\lambda_{o1} - \lambda_{o2}}{\lambda_{o1} - {}^{1}\lambda_{o1}} \qquad \ldots (4.61)$$

Therefore,

$${}^{1}\lambda_{o1} = \lambda_{o2} + \frac{(\lambda_{o1} - \lambda_{o2})\times(\lambda_{o3} - \lambda_{o2})}{(\lambda_{o1} - \lambda_{o2})+(\lambda_{o3} - \lambda_{o2})} \qquad \ldots (4.62)$$

Computer calculations show that the error (as plotted in Fig. 4.

$$\mathcal{E}_{\lambda} = \frac{{}^{1}\lambda_{o1} - \lambda_{oe}}{\lambda_{oe}} \leqslant 1.5\ \% \qquad \ldots (4.63)$$

over the entire range $0.1 < n < 5$ if λ_{o1} and $f(\lambda)$ are chosen according to Eqns. (4.59) and **Fig.4.6** respectively. Hence,

$$\lambda_{o} \simeq {}^{1}\lambda_{o1} \qquad \ldots (4.64)$$

Sometimes it is not necessary to calculate all the 4 values viz. λ_{o1}, λ_{o2}, λ_{o3} and ${}^{1}\lambda_{o1}$. For example, suppose $\left(\lambda_{o1} \sim \lambda_{o2}\right) < 5\ \%$. Now λ_{oe} lies between these two values so that $(\lambda_{o1} \sim \lambda_{oe})/\lambda_{oe} < 5\ \%$ and $(\lambda_{o2} \sim \lambda_{oe})/\lambda_{oe} < 5\ \%$. Then one may write

$$\lambda_{o} \simeq \lambda_{o1} \simeq \lambda_{o2} \qquad \ldots (4.65)$$

$$\text{for } \left(\lambda_{o1} \sim \lambda_{o2}\right) < 5\ \%$$

and it is not necessary to calculate λ_{o3} and ${}^{1}\lambda_{o1}$. Similarly one may avoid the calculation of ${}^{1}\lambda_{o1}$ and write

$$\lambda_{o} \simeq \lambda_{o2} \simeq \lambda_{o3} \qquad \ldots (4.66)$$

$$\text{for } \left(\lambda_{o2} \sim \lambda_{o3}\right) < 5\ \%$$

The calculation of λ_o for the range $0.1 < n < 5$ is summarized below:

$$\left.\begin{aligned}
&\lambda_{o1} = \left(\frac{\sin\theta}{n}\right)^{\frac{1}{3}}, \quad \lambda_{o2} = f(\lambda_{o1}) \text{ and } \lambda_{o3} = f(\lambda_{o2}) \\
&\text{where,} \\
&f(\lambda) = f_1(\lambda) = \frac{\frac{\sin\theta}{n}}{1+\frac{\cos\theta}{n} + \lambda^2} \quad \text{for } n > -\cos\theta \\
&f(\lambda) = f_2(\lambda) = \sqrt{\frac{\sin\theta}{n\lambda} - \left(1 + \frac{\cos\theta}{n}\right)} \quad \text{for } n < -\cos\theta \\
&\lambda_o \simeq \lambda_{o2} + \frac{(\lambda_{o1} - \lambda_{o2})\times(\lambda_{o3} - \lambda_{o2})}{(\lambda_{o1} - \lambda_{o2})+(\lambda_{o3} - \lambda_{o2})}
\end{aligned}\right]$$

The error in λ_o as calculated above is plotted as a function of θ in Fig. 4.9.

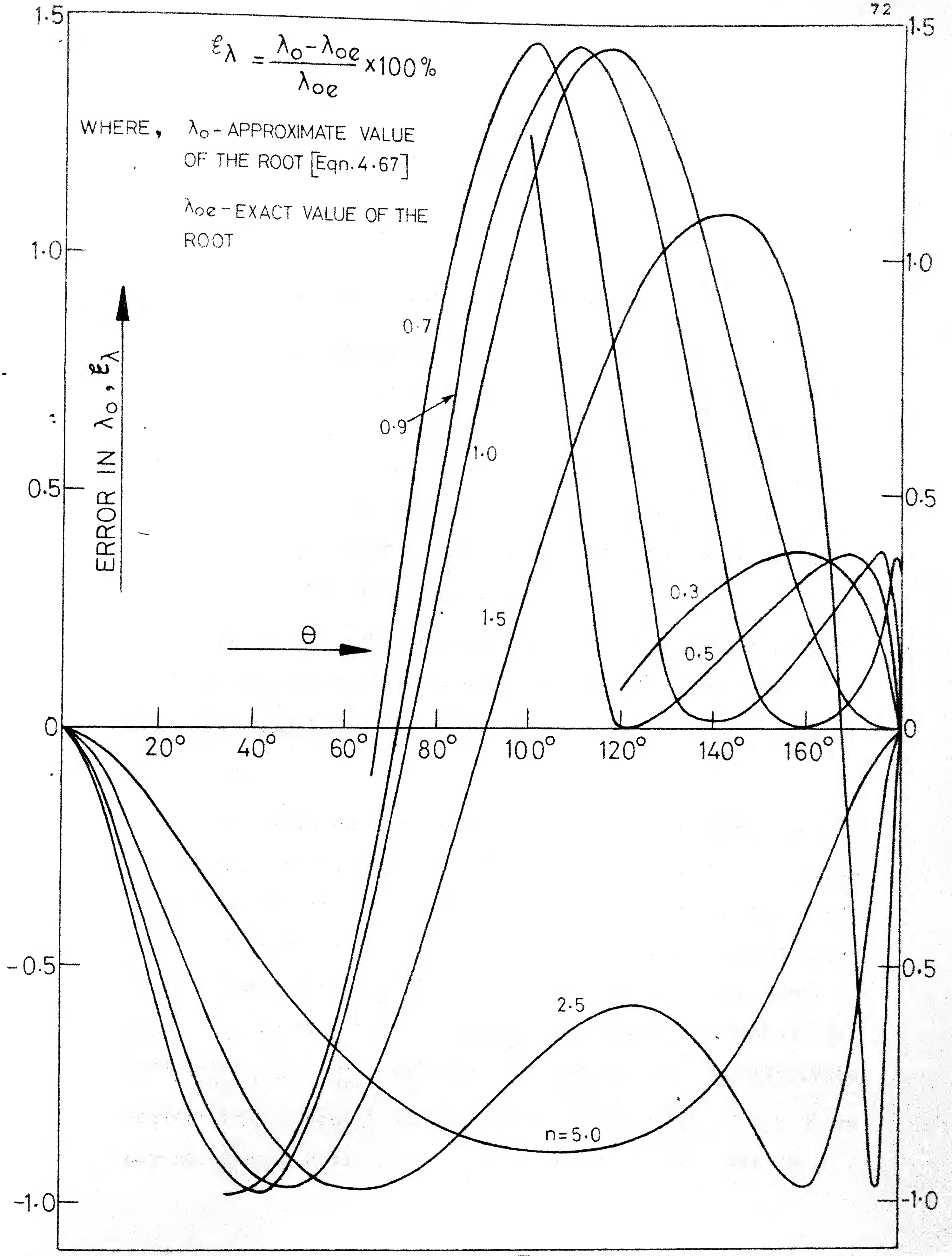


FIG. 4·9 VARIATION OF ERROR IN $\lambda_o, \varepsilon_\lambda$ [Eqn. 4·67] WITH θ FOR VARIOUS

CHAPTER 5

ERRORS IN TERMINATIONS AND GAIN

5.1 Error in Termination

Equations (3.21) and (3.22) show that the real parts of the source and load terminations ρ_S, ρ_L are independent of λ_o so that the error in the real parts of terminations is nil. However, the imaginary parts, σ_S, σ_L depend upon λ_o as per Eqn. (3.27). The error in the imaginary parts is investigated in Appendix L.

It is shown that the modulus of the error in σ, $|\mathcal{E}_\sigma|$ is less than the modulus of error in λ, $|\mathcal{E}_\lambda|$, provided $\left(\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\ \frac{\sigma_{11}}{\rho_{11}}\right)$ and $\left(\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\ \frac{\sigma_{22}}{\rho_{22}}\right)$ are both < 0 or are both > 2.

For junction transistors in their common emitter configuration and y-matrix environment or in the common base configuration and h-matrix environment, over the frequency range of potential instability, where design of mismatched stages may be necessary, arg $(p_{12}p_{21})$ is usually between -0° and -180° with σ_{11} and σ_{22} both positive. This makes λ_o negative so that $\left(\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\ \frac{\sigma_{11}}{\rho_{11}}\right)$ and $\left(\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\ \frac{\sigma_{22}}{\rho_{22}}\right)$ are both negative. This corresponds to case 1 of Appendix F. Since $|\mathcal{E}_\lambda| < 1.5\ \%$ as per sections 4.2 to 4.4, $|\mathcal{E}_\sigma|$ is also $< 1.5\ \%$. For the

same device in the common-base configuration over the frequency range of potential instability, arg $(p_{12}p_{21})$ is between 0° and 180° for y-matrix environment. Here, λ_o is positive. For common-emitter configuration, arg $(p_{12}p_{21})$, over the same frequency range is between -0° and -180° for h-matrix environment. Here, λ_o is negative. For both situations either σ_{11} or σ_{22} is positive. Thus either $\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\frac{\sigma_{11}}{\rho_{11}}$ or $\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\frac{\sigma_{22}}{\rho_{22}}$ is positive and corresponds to case 2 or case 3 of Appendix L. There are possibilities corresponding to case 3 with $\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\frac{\sigma_{11}}{\rho_{11}}$ or $\frac{1}{\lambda_o}\sqrt{\frac{n_i}{n}}\frac{\sigma_{22}}{\rho_{22}}$ positive and between 0 and 2, so that $|\mathcal{E}_{\sigma}|$ may be $>|\mathcal{E}_{\lambda}|$. This is confirmed by calculations on typical devices. It is now appropriate to consider how these errors in termination affect the error in power gain.

5.2 Error in Maximum Power Gain Due to Approximate Root:

The expression for maximum operating power gain is repeated below (Eqns. 3.25 and 3.26)

$$g_{max\ n} = 4\ F\left|\frac{p_{21}}{p_{12}}\right|\left\{1-\sqrt{\frac{n_i}{n}}\right\}^2 \quad \ldots \quad (5.1)$$

where,

$$F = \frac{n}{\text{Denom.}} = \frac{n}{\left[n(1-\lambda_o^2) - \cos\theta\right]^2 + \left[2n\lambda_o - \sin\theta\right]^2} \quad \ldots \quad (5.2)$$

The errors in F and hence in $g_{max\ n}$ for each of the separate ranges of performance factor values, can now be considered.

5.2.1 Gain Error, When $n \geqslant 5$

In this case, the first approximate root of Eqn.(4.31) is substituted for λ_o. It was shown in Section 4.2 that the error in this approximation is $\leqslant 0.041$. Due to the error in λ_o, the calculated value of F will also be in error. It is shown in Appendix G that this error is < 0.028 %. On substituting for λ_o, F becomes,

$$F = \frac{n}{\left[n-n\lambda_o^2 - \cos\theta\right]^2 + \left[2n\lambda_o - \sin\theta\right]^2} \quad \ldots (5.3)$$

$$= \frac{n}{\left[n-\cos\theta - \frac{n\sin^2\theta}{(n+\cos\theta)^2}\right]^2 + \left[\frac{2n\sin\theta}{n+\cos\theta} - \sin\theta\right]^2} \quad \ldots (5.4)$$

It is shown in Appendix H that this expression can be simplified to

$$F = \frac{n}{n^2-1}\left(\frac{n+\cos\theta}{n-\cos\theta}\right) \quad \ldots (5.5)$$

with an error of < 0.27 %. Thus the overall error in expression (5.3) is $\simeq < 0.27$ %. Actual computer calculations show that the error in expression (5.5) is $\leqslant 0.15$ %. Thus the maximum gain is given by

$$g_{max\ n} = \frac{4n}{n^2-1}\left(\frac{n+\cos\theta}{n-\cos\theta}\right)\left|\frac{p_{21}}{p_{12}}\right|\left\{1-\sqrt{\frac{n_i}{n}}\right\}^2 \quad (5.6)$$

within 0.15 % ($\simeq$ 0.01 dB)

5.2.2 Gain Error, When $n \leqslant 0.1$

The final approximate root of Eqn. (4.46) gives a power gain with an error modulus as indicated by Fig. 5.1, which is based on computer calculations. Computer calculations show that the error is < 0.1 % (< ≃ 0.0045 dB).

5.2.3 Gain Error, When $0.1 \leqslant n \leqslant 5$ with $n \geqslant 1.3\,\Pi$

The final approximate root of Eqn. (4.67) gives a power gain with an error modulus as indicated by Fig. 5.1 which is based on computer calculations. Provided $n \geqslant 1.3\,\Pi$ or $1.3\left(\frac{1+\cos\theta}{2}\right)$, the modulus of error is < 3 % (< ≃ 0.13 dB).

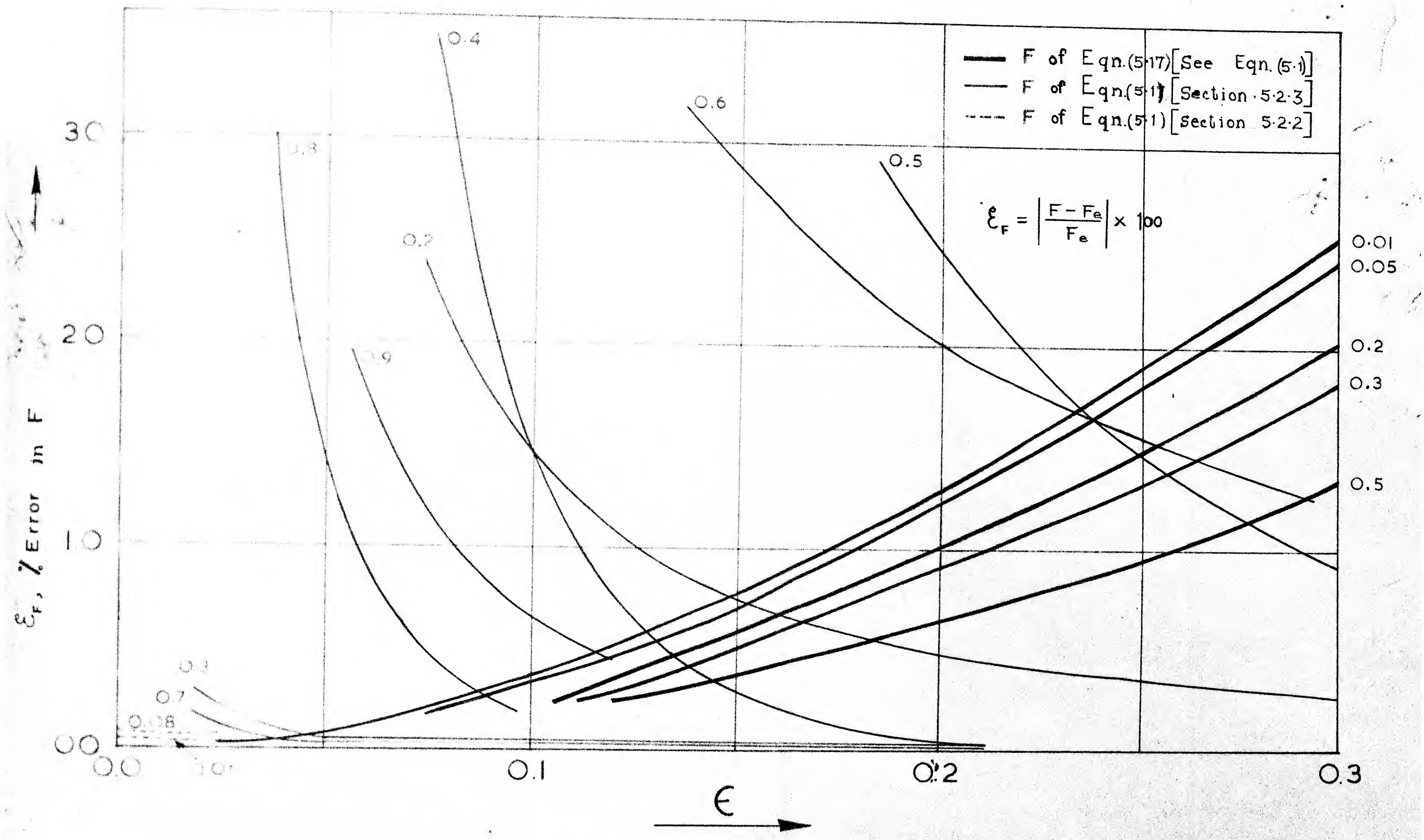
5.3 Maximum Power Gain, When $n < 1.3\,\Pi$

When n is near Π or $\left(\frac{1+\cos\theta}{2}\right)$, the maximum power gain of the two-port network is large and sensitive to termination or λ_o. Even though error in λ_o or error in termination is small, error in power gain can be very large. The calculated value of F can be written from Eqn. (G-3) of Appendix G as

$$F = \frac{n}{(\mathrm{Denom})_e + 2n\lambda_{oe}^2\,\varepsilon_\lambda^2\,(n + 3n\lambda_{oe}^2 + \cos\theta)} \tag{5.7}$$

Here, $(\mathrm{Denom})_e$ is the exact value of the denominator of F and ε_λ is the error in λ. When n is very close to Π or $\left(\frac{1+\cos\theta}{2}\right)$, the root is given by[8]

$$\lambda_o \simeq \frac{N}{L+M} = \tan\left(\frac{\theta}{2}\right) \qquad \ldots (5.8)$$


F of Eqn.(5·17) [See Eqn. (5·1)]
F of Eqn.(5·1) [Section 5·2·3]
F of Eqn.(5·1) [Section 5·2·2]
$\varepsilon_F = \left| \frac{F - F_e}{F_e} \right| \times 100$
ε_F, % Error in F
ϵ
0.0
0.1
0.2
0.3
10
20
30
0.4
0.6
0.5
0.8
0.2
0.9
0.3
0.7
0.08
0.01
0.05
0.2
0.3
0.5

Then, the error term in F becomes,

$$2\cos^2(\tfrac{\theta}{2})\ \tan^2(\tfrac{\theta}{2})\ \left[\cos^2(\tfrac{\theta}{2}) + 3\ \cos^2(\tfrac{\theta}{2})\ \tan^2(\tfrac{\theta}{2}) + 2\cos^2(\tfrac{\theta}{2}) - 1\right] \mathcal{E}_\lambda^{\ 2} \qquad \ldots (5.9)$$

$$= 4\ \sin^2(\tfrac{\theta}{2})\ \mathcal{E}_\lambda^{\ 2} \qquad \ldots (5.10)$$

Also,

$$(\text{Denom})_e \simeq \cos^2(\tfrac{\theta}{2})\epsilon^2 \qquad \ldots (5.11)$$

where

$$n = \Pi\ (1 + \epsilon) \qquad \ldots (5.12)$$

Thus, for n very close to Π or $\epsilon << 1$,

$$F \simeq \frac{n}{\cos^2(\frac{\theta}{2})\epsilon^2 + 4\ \sin^2(\frac{\theta}{2})\ \mathcal{E}_\lambda^{\ 2}} \qquad \ldots (5.13)$$

$$= \frac{n}{\Pi\ \epsilon^2 + 4(1-\Pi)\ \mathcal{E}_\lambda^{\ 2}} \qquad \ldots (5.14)$$

In the range $0.1 < n < 5$, the error $\mathcal{E}_\lambda$ tends to some non-zero value as $\epsilon \to 0$ so that the calculated value of F is finite though the actual value $\frac{n}{\Pi\ \epsilon^2} \to \infty$. The error in gain therefore approaches 100 %. The error is kept below 3 % by restricting the use of λ_o of Eqn. (4.67) to calculate the gain, to $\epsilon \geqslant 0.3$ or $n \geqslant 1.3\ \Pi$.

In the range $n \leqslant 0.1$, however, the error $\mathcal{E}_\lambda$ in λ_o, as per Eqn. (4.46), approaches zero as $\epsilon \to 0$, in such a way that (Appendix I)

$$\underset{\epsilon \to 0}{\text{Lt}} \left(\frac{1-\Pi}{\Pi} \times \frac{4\ \mathcal{E}_\lambda^{\ 2}}{\epsilon^2} \right) \quad \text{is} \quad < 0.00137 \qquad \ldots (5.15)$$

$$\text{for } n \leqslant 0.1$$

Hence the error in F does not approach 100% but about 0.137% as $\epsilon \to 0$.

Hence it should seem necessary to evaluate an equally accurate root in the range $0.1 < n < 5$, so that the gain error is small.

5.3.1 λ_o, When $n \leqslant 1.3\pi$

When $n = \pi$, the root is given by Eqn. (5.8). It is now possible to expand λ_o around the value $\tan(\frac{\theta}{2})$, corresponding to $n = \pi$. This Taylor's series expansion is derived in Appendix J. Thus,

$$\lambda_o \simeq \left[1 - \frac{\epsilon}{2} + \frac{\epsilon^2}{8}(3 - \pi)\right] \tan\left(\frac{\theta}{2}\right) \qquad \ldots (5.16)$$

with an error < 1 %. Computer calculations show that the error is $\leqslant$ 0.71 %.

When only two terms are taken, $\lambda_o \simeq (1 - \frac{\epsilon}{2}) \tan(\frac{\theta}{2})$ as for the case where $\epsilon \ll 1$ corresponding to the gain margin discussed previously.[8]

5.3.2 Maximum Power Gain, When $n \leqslant 1.3\pi$

The maximum power gain can be obtained from Eqns. (5.1) and (5.2) after substituting for λ_o from Eqn. (5.16). It is shown in Appendix K that,

$$g_{max\ n} = \frac{4}{\epsilon^2}\left[1 + \frac{3-\pi}{4}\epsilon\right]\left|\frac{p_{21}}{p_{12}}\right|\left\{1 - \sqrt{\frac{n_i}{n}}\right\}^2 \qquad (5.17)$$

$n \leqslant 1.3\pi$

where

$$n = \prod (1 + \epsilon)$$

provided $\prod \neq 0$ or $\theta \neq 180^{\circ}$. It is estimated in the same Appendix that the error modulus in the above power gain is $< 3\,\%$. Computer calculations show that the error is $<3\,\%$. (Fig. 5.1)

CHAPTER 6

SUMMARY AND ILLUSTRATIVE EXAMPLE

6.1 Summary:

In Chapter 4 approximate value of λ_o can be calculated from Eqns. (4.31), (4.32), (4.45), (4.46) and (4.67) which are repeated below. While calculating λ_o from the following expressions θ is taken to be positive. The actual value of λ_o is then given by Eqn. (3.28) viz.

$$\lambda_o(-\theta) = -\lambda_o(\theta) \qquad \ldots(6.1)$$

Case 1: $n \geqslant 5$

$$\lambda_o \simeq \frac{\sin\theta}{n + \cos\theta} \quad \text{within } 4.1\% \qquad \ldots(6.2)$$

$$\lambda_o \simeq \frac{\sin\theta}{n + \cos\theta}\left[1 - \frac{n\sin^2\theta}{(n + \cos\theta)^3}\right] \quad \text{within } 0.5\% \qquad \ldots(6.3)$$

Case 2: $0.1 \leqslant n \leqslant 5$

$$\left.\begin{aligned} \lambda_o &\simeq \lambda_{o2} + \frac{(\lambda_{o1} - \lambda_{o2}) \times (\lambda_{o3} - \lambda_{o2})}{(\lambda_{o1} - \lambda_{o2}) + (\lambda_{o3} - \lambda_{o2})} \quad \text{within } 1.5\% \\ \text{where} \quad \lambda_{o1} &= \left(\frac{\sin\theta}{n}\right)^{\frac{1}{3}} \\ \lambda_{o2} &= f(\lambda_{o1}) \\ \lambda_{o3} &= f(\lambda_{o2}) \end{aligned}\right\} \qquad (6.4)$$

and

$$f(\lambda) = f_1(\lambda) = \frac{\frac{\sin\theta}{n}}{1 + \frac{\cos\theta}{n} + \lambda^2} \quad \text{for } n+\cos\theta > 0$$

$$f(\lambda) = f_2(\lambda) = \sqrt{\frac{\frac{\sin\theta}{n}}{\lambda} - \left(1 + \frac{\cos\theta}{n}\right)} \quad \text{for } n + \cos < 0 \qquad (6.4$$

Case 3: $n \leqslant 0.1$

$$\lambda_o \simeq \sqrt{\frac{1-n}{n}} \quad \text{within } 3\% \qquad \ldots(6.5)$$

$$\lambda_o \simeq \sqrt{\frac{1-n}{n}} \left\{1 - \frac{1}{2(1-n)}\left[\frac{\sin\theta}{\sqrt{\frac{1-n}{n}}} - (1+\cos\theta)\right]\right\} \quad \text{within } 0.33\% \qquad \ldots(6.6)$$

Once λ_o is known the terminations can be found as (Eqn. 3.21, 3.22 and 3.27)

$$\rho_S = \rho_{11}\left\{\sqrt{\frac{n}{n_i}} - 1\right\}$$

$$\rho_L = \rho_{22}\left\{\sqrt{\frac{n}{n_i}} - 1\right\} \qquad \ldots(6.7)$$

$$\sigma_S = \lambda_o\,(\rho_{11} + \rho_S) - \sigma_{11}$$

$$\sigma_L = \lambda_o\,(\rho_{22} + \rho_L) - \sigma_{22} \qquad \ldots(6.8)$$

Because of the error in λ_o the values of σ_S, σ_L will also be in error. This error is investigated in Appendix L.

Next, the value of maximum operating power gain is calculated from expressions (5.1), (5.6) or (5.17). Thus

Case 1: $n \geqslant 5$

$$g_{max\ n} = 4 \frac{n}{n^2-1} \left(\frac{n+\cos\theta}{n-\cos\theta}\right) \left|\frac{p_{21}}{p_{12}}\right| \left\{1-\sqrt{\frac{n_i}{n}}\right\}^2 \quad ..(6.9)$$

within 0.15% ($\simeq$ 0.007 dB) (6.9)

Case 2: $n \leqslant 5$ but $\geqslant 1.3\,\pi$

$$g_{max\ n} = 4 \frac{n}{\left[n(1-\lambda_o^2)-\cos\theta\right]^2 + \left[2n\lambda_o - \sin\theta\right]^2} \left|\frac{p_{21}}{p_{12}}\right| \left\{1-\sqrt{\frac{n_i}{n}}\right\}^2$$

within 3% ($\simeq$ 0.13 dB) ...(6.10)

Case 3: $n \leqslant 1.3\,\pi$

$$g_{max\ n} = \frac{4}{\epsilon^2} \left[1 + \frac{3-\pi}{4}\epsilon\right] \left|\frac{p_{21}}{p_{12}}\right| \left\{1-\sqrt{\frac{n_i}{n}}\right\}^2 \quad ...\ (6.11)$$

where $n = \pi\,(1+\epsilon)$ within 3% ($\simeq$0.13 dB)

6.2 Illustrative Example:

Consider a transistor type AF117 operating at 455 kHz in the common-emitter configuration for an emitter current of 2mA, collector-base voltage of -6V and an ambient temperature of 298^oK. With biasing, a typical set of measured admittance parameters are[12]

$$y_{11} = (510 + j410)10^{-6}\Omega^{-1}$$

$$y_{12} = -j8 \times 10^{-6}\,\Omega^{-1}$$

$$y_{21} = 70 \times 10^{-3}\,\Omega^{-1}$$

and

$$y_{22} = (2.5 + j14)10^{-6}\,\Omega^{-1}$$

$$\therefore \quad y_{12}\, y_{21} = 0.56 \times 10^{-6} \;\angle -90^{\circ}\;\Omega^{-2}$$

$$n_i = \frac{\rho_{11}\,\rho_{22}}{L} = \frac{0.51 \times 2.5 \times 10^{-9}}{0.56\times 10^{-6}} = 2.28 \times 10^{-3}$$

$$\Pi = \frac{1 + \cos\theta}{n} = \frac{1 + \cos(-90^{\circ})}{2} = 0.5$$

Let us design a mismatched amplifier stage for $n = 2$. Using Eqns. (6.1) and (6.4) ($f_1(\lambda)$ is used as $n + \cos\theta > 0$)

$$\lambda_o = -0.42$$

From Eqns. (6.7) and (6.8) the source and load terminations are obtained

$$\rho_S = 14.6\ \mathrm{m}\Omega^{-1}$$

$$\sigma_S = -6.76\ \mathrm{m}\Omega^{-1} \quad \therefore L_S^{\ddagger} = 51.8\,\mu\mathrm{H} \text{ or } L_S^{\S} = 51.5\,\mu\mathrm{H}$$

$$\rho_L = 71.5\,\mu\Omega^{-1}$$

$$\sigma_L = -44.4\,\mu\Omega^{-1} \quad \therefore L_L^{\ddagger} = 7.9\mathrm{mH} \text{ or } L_L^{\S} = 4.89\mathrm{mH}$$

As $n > 1.3\,\Pi$, Eqn. (6.10) is used to calculate the gain

---

‡Without any coil or stray capacitance.

§ With a realistic value of 10 pF for the total (coil + stray) capacitance.

$$g_{max\ n} = 43.\ \text{dB}$$

The mismatched stage has been designed with the reactances tuned to 455 kHz. These terminations are shown in Fig. 6.1, assuming a value of 10 pF for the total (coil + stray) capacitance ~~x~~ at each port.

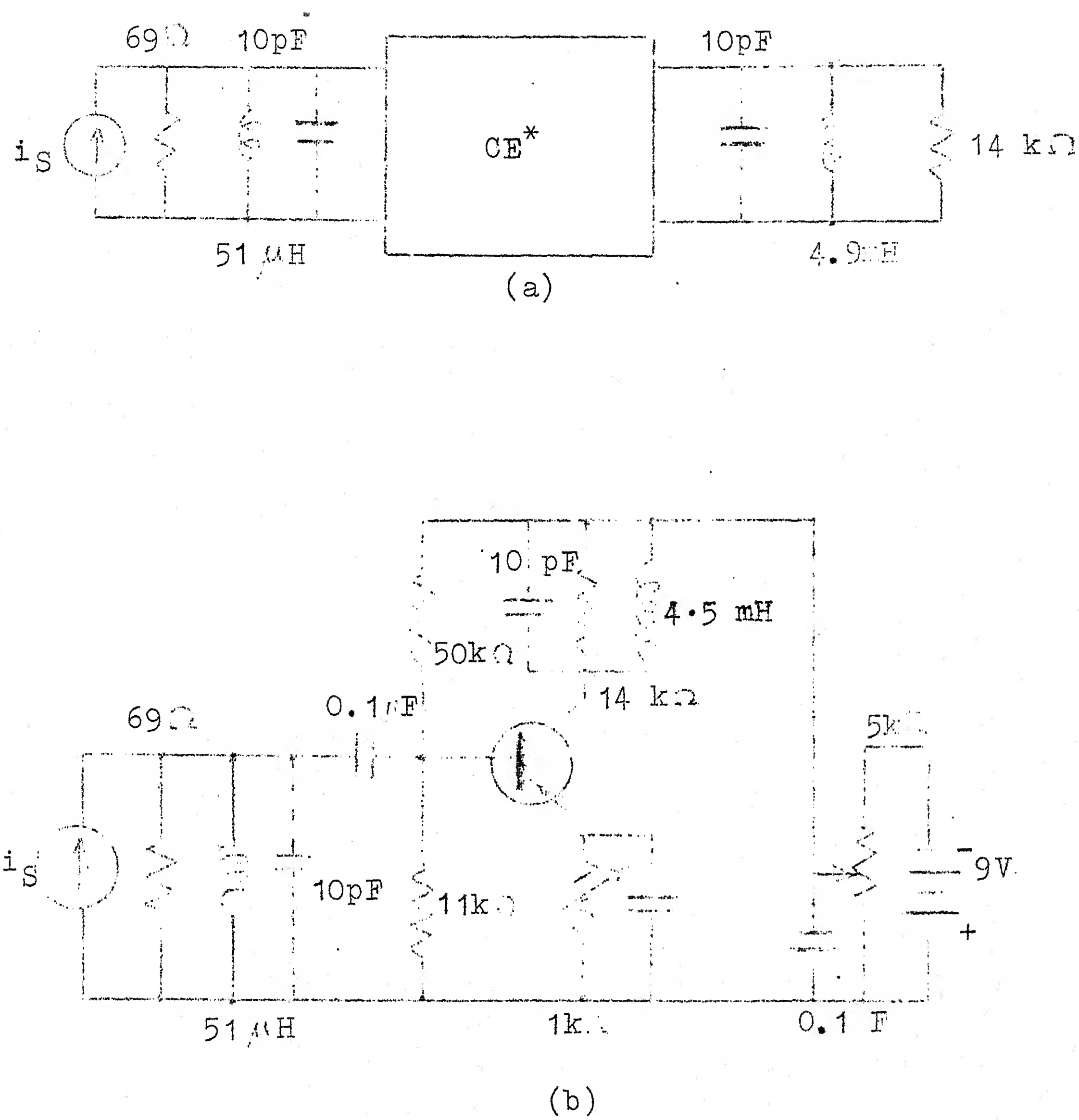


Fig. 6.1 (a) Terminations for a performance factor of 2 in the common-emitter configuration and y-matrix environment.

(b) Complete circuit diagram of the mismatched amplifier stage. Total capacitance (coil+stray) is shown by dotted lines. Device used is a transistor type AF117 at $I_e = 2mA, V_{cb} = -6V$ at 298^oK.

* Measured admittance parameters of 'black box' include the biasing resistances.

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APPENDIX A

Substitution of λ_1, λ_2 and λ_3, the three real roots of cubic equation (4.1) in succession for λ_o of Eqn. (3.26) gives the three possible power gains. Of these, only one gives the maximum power gain. Call the denominator of F of Eqn.(3.26), d_1 when λ_o is replaced by λ_1 etc. Then

$$d_1 = \left[n(1-\lambda_1^2) - \cos\theta\right]^2 + \left[2n\lambda_1 - \sin\theta\right]^2 \quad (A-1)$$

$$= n^2(1-2\lambda_1^2+\lambda_1^4) - 2n\cos\theta(1-\lambda_1^2) + \cos^2\theta$$
$$+4n^2\lambda_1^2 - 4n\lambda_1\sin\theta + \sin^2\theta \quad (A-2)$$

$$= n^2\lambda_1^4 + 2n^2\lambda_1^2 + 2n\cos\theta\,\lambda_1^2 - 4n\lambda_1\sin\theta$$
$$+ 1-2n\cos\theta + n^2 \quad (A-3)$$

$$= n^2\lambda_1\left[\lambda_1^3 + \left(1+\frac{\cos\theta}{n}\right)\lambda_1 - \frac{\sin\theta}{n}\right]$$
$$+ n^2\left[\left(1+\frac{\cos\theta}{n}\right)\lambda_1^2 - 3\left(\frac{\sin\theta}{n}\right)\lambda_1\right]$$
$$+ 1 - 2n\cos\theta + n^2 \quad (A-4)$$

$$= n^2\lambda_1(\lambda_1^3 - b\lambda_1 - a) - n^2(b\lambda_1^2 + 3a\lambda_1)$$
$$+ 1 - 2n\cos\theta + n^2 \quad (A-5)$$

As λ_1 is a root of the cubic eqn. (4.1),

$$\lambda_1^3 - b\lambda_1 - a = 0 \quad (A-6)$$

$$\therefore\quad d_1 = -n^2(b\lambda_1^2 + 3a\lambda_1) + 1-2n\cos\theta + n^2 \quad (A-7)$$

Similarly,

$$d_2 = -n^2(b\lambda_2^2 + 3a\lambda_2) + 1-2n\cos\theta + n^2 \qquad \ldots\text{(A-8)}$$

$$d_3 = -n^2(b\lambda_3^2 + 3a\lambda_3) + 1-2n\cos\theta + n^2 \qquad \ldots\text{(A-9)}$$

Hence,

$$d_1 - d_2 = -n^2(\lambda_1 - \lambda_2)\left[b(\lambda_1 + \lambda_2) + 3a\right] \qquad \ldots\text{(A-10)}$$

$$d_1 - d_3 = -n^2(\lambda_1 - \lambda_3)\left[b(\lambda_1 + \lambda_3) + 3a\right] \qquad \ldots\text{(A-11)}$$

It is shown in Section 4.1 that b must be positive for the case of 3 real roots. If the case of $a = \frac{\sin\theta}{n} > 0$ is considered, then it follows from Section 4.1, that,

$$\begin{array}{ll} \lambda_1 - \lambda_2 > 0, & \lambda_1 + \lambda_2 > 0 \\ \lambda_1 - \lambda_3 > 0, & \lambda_1 + \lambda_3 > 0 \end{array} \qquad \ldots\text{(A-12)}$$

so that,

$$d_1 - d_2 < 0, \quad d_1 - d_3 < 0 \qquad \ldots\text{(A-13)}$$

Thus d_1 is smaller then d_2 or d_3. Hence λ_o, the root that gives optimum power gain, equals the only positive root λ_1.

If the case of $a = \frac{\sin\theta}{n} < 0$ is considered, then,

$$\begin{array}{ll} \lambda_1 - \lambda_2 < 0, & \lambda_1 + \lambda_2 < 0 \\ \lambda_1 - \lambda_3 < 0, & \lambda_1 + \lambda_3 < 0 \end{array} \qquad \ldots\text{(A-14)}$$

where λ_1 is the negative root and λ_2, λ_3 the positive roots. Therefore, the inequalities (A-13) follow once again from Eqns. (A-10) and (A-11). This shows that for the case of a < 0, λ_o equals the only negative root λ_1.

The same conclusions follow graphically. Fig. A-1(a) shows the variation of F of eqn. (3.26) with λ for the case of a > 0 and a single positive real root λ_1. Fig. A-1(b) shows the same for the case of a > 0 and three real roots. It is easily seen that the only positive root λ_1 gives the maximum value for F. Similarly, Fig. A-2(b) shows that the only negative root λ_1 gives the maximum value of F.

APPENDIX B

Consider Eqn. (4.20)

$$\lambda = \frac{a}{c + \lambda^2} \qquad \text{... (B-1)}$$

where

$$a = \frac{\sin\theta}{n}, \qquad c = 1 + \frac{\cos\theta}{n} \qquad \text{... (B-2)}$$

Then,

$$|\lambda| \leqslant \left|\frac{a}{c}\right| \qquad \text{... (B-3)}$$

provided that $c > 0$ and $\lambda^2 > 0$. The former inequality is true for $n \geqslant 1$ and the latter for real λ. Then from Eqn.(4.22)

$$|X_1| = \left|\frac{\lambda^3}{a}\right| < \frac{a^2}{c^3} = T \qquad \text{... (B-4)}$$

using Eqn.(4.26)

The modulus sign is not used for $T = \frac{a^2}{c^3}$ as T is a positive quantity for $n \geqslant 1$. Hence,

$$|X_1| \ll 1 \quad \text{for } T \ll 1 \qquad \text{... (B-5)}$$

Again, consider Eqn. (4.25)

$$X_2 = \frac{a\,\lambda^3}{c^3}\left(3 - \frac{3\,\lambda^3}{a} + \frac{\lambda^6}{a^2}\right) \qquad \text{... (B-6)}$$

$$= \frac{3a\,\lambda^3}{c^3}\left(1 - \frac{\lambda^3}{a} + \frac{\lambda^6}{3a^2}\right) \qquad \text{... (B-7)}$$

The second and third terms in the parenthesis have magnitudes less than T and $T^2/3$ respectively and may be neglected for $T \ll 1$.

Then

$$|X_2| \simeq \frac{3a\,\lambda^3}{c^3} \leqslant 3\,T^2 \qquad \ldots \text{(B-8)}$$

Hence,

$$|X_2| \ll 1 \quad \text{for } T \ll 1 \qquad \ldots \text{(B-9)}$$

Similarly it may be shown that

$$|X_3| \ll 1\,, \; |X_4| \ll 1 \text{ etc. for } T \ll 1 \qquad \text{(B-10)}$$

APPENDIX C

Consider Eqn. (4.26)

$$T = \frac{a^2}{c^3} = \frac{n \sin^2\theta}{(n + \cos\theta)^3} \qquad \ldots \quad (C-1)$$

For a given $n > 1$, T is positive and has a maximum value found as below,

$$\frac{\partial T}{\partial \theta} = 0 \text{ gives,}$$

$$\cos\theta = n - \sqrt{n^2 + 3} \qquad \ldots \quad (C-2)$$

Therefore,

$$T_{\max n} = \frac{2n\left[n\sqrt{n^2+3} - n^2 - 1\right]}{\left[2n - \sqrt{n^2+3}\right]^3} \qquad \ldots \quad (C-3)$$

This is plotted as a function of n in Fig. C-1 which shows that the function decreases monotonically as n increases.

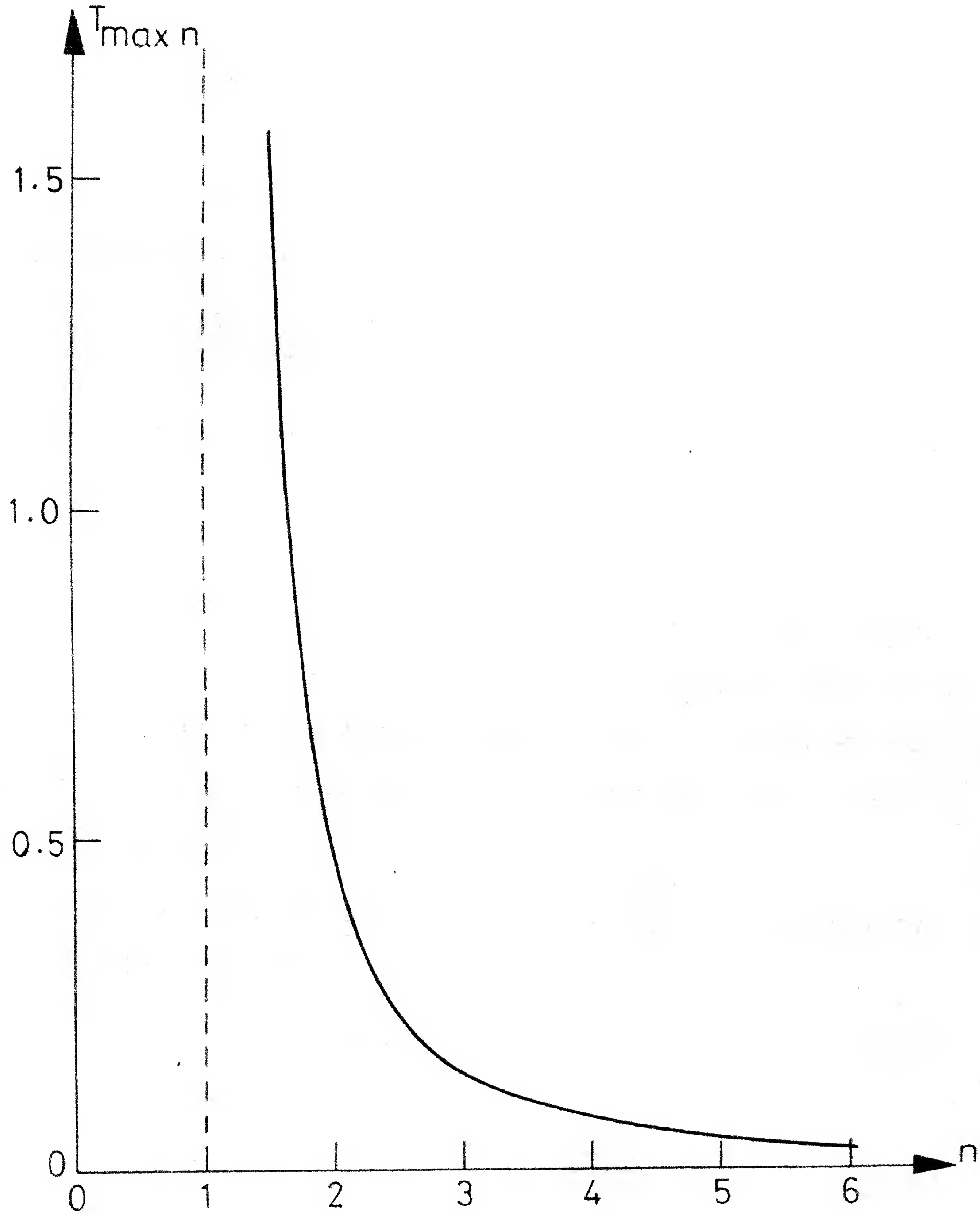

FIG. C-1 VARIATION OF $T_{max\,n}$ OF Eqn.(C-3) WITH n

APPENDIX D

Consider the Eqns. (4.36), (4.38) and (4.44)a which are repeated below:

$$\lambda_o = \sqrt{\frac{1-n}{n}}\sqrt{1+A} \tag{D-1}$$

$$A = \frac{1}{1-n}\left[\frac{\sin\theta}{\lambda_o} - (1+\cos\theta)\right] \tag{D-2}$$

$$\mathcal{E}_\lambda = \frac{1}{\sqrt{1+A}} - 1 \tag{D-3}$$

$|\mathcal{E}_\lambda|$ increases monotonically with A as shown in Fig. D-1. Also it will be shown below that the maximum value of A for a given n increases monotonically with n as shown in Fig. D-3. Thus if $|\mathcal{E}_\lambda|$ is to be $\leqslant 0.05$, then A must be $\leqslant 0.108$ and hence $n \leqslant 0.16$.

First it will be shown that $\lambda_o \geqslant \sqrt{\frac{1-n}{n}}$. Consider the function $F(\lambda)$ given by,

$$F(\lambda) = \lambda^3 + c\lambda - a \tag{D-4}$$

where,

$$c = 1 + \frac{\cos\theta}{n} \quad ; \quad a = \frac{\sin\theta}{n} \tag{D-5}$$

The zeros of $F(\lambda)$ correspond to the roots of the cubic equation (4.17) $F(\lambda)$ is plotted in Fig. D-2a for the case $a > 0$. In the figures, λ_o is shown to be the single positive root, the other two being complex or both negative. Now

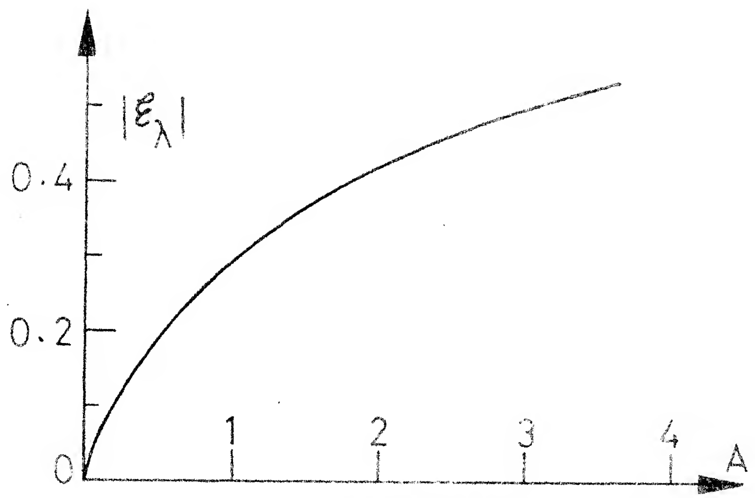

FIG. D-1 VARIATION OF ERROR MODULUS IN λ $|\varepsilon_\lambda|$ WITH A

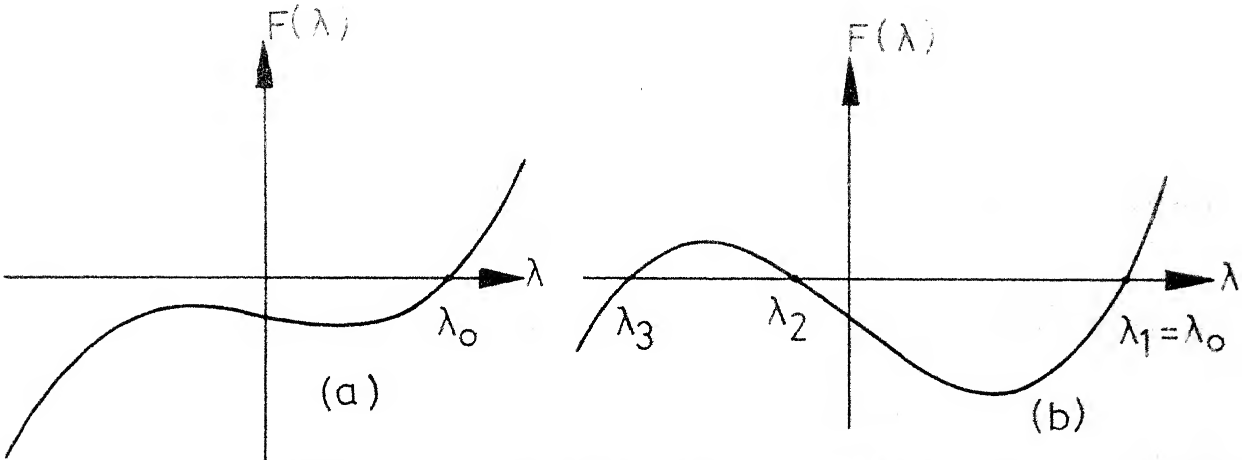

FIG. D-2 VARIATION OF $F(\lambda)$ OF Eqn.(D-4) WITH λ FOR THE CASES OF A SINGLE REAL ROOT & 3 REAL ROOTS RESPECTIVELY

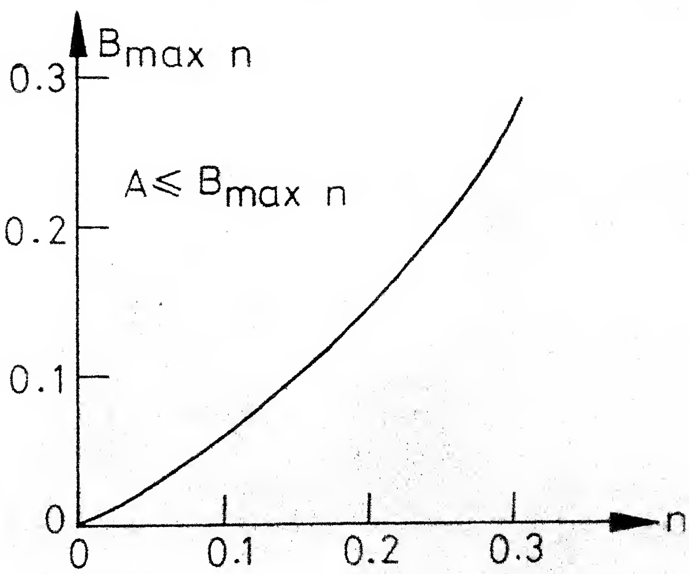

FIG. D-3 VARIATION OF B_{max} n OF Eqn.(D-10) WITH n

$$F\left(\sqrt{\frac{1-n}{n}}\right) = \left(\sqrt{\frac{1-n}{n}}\right)^3 + \left(1 + \frac{\cos\theta}{n}\right)\sqrt{\frac{1-n}{n}} - \frac{\sin\theta}{n} \qquad (D\text{-}6)$$

$$= \frac{1}{n}\left[\sqrt{\frac{1-n}{n}}\,(1+\cos\theta) - \sin\theta\right] \qquad (D\text{-}7)$$

which is $\leqslant 0$ in the absolutely stable region corresponding* to $n > \prod = \frac{1+\cos\theta}{2}$. From the figures D-2, it is seen that $F\left(\sqrt{\frac{1-n}{n}}\right)$ can be $\leqslant 0$ only if $\sqrt{\frac{1-n}{n}} \leqslant \lambda_o$. Thus,

$$\lambda_o \geqslant \sqrt{\frac{1-n}{n}}$$

$$\therefore A \geqslant 0 \qquad (D\text{-}8)$$

Then it follows from Eqns. (D-2) and (D-8) that

$$0 \leqslant A \leqslant \frac{1}{(1-n)}\left[\frac{\sin\theta}{\sqrt{\frac{1-n}{n}}} - (1+\cos\theta)\right] = B \qquad (D\text{-}9)$$

The maximum of B for a given n occurs at $\cos\theta = -\sqrt{1-n}$ i.e. $\sin\theta = \sqrt{n}$. Thus,

$$0 \leqslant A < B_{max\ n} = \frac{1}{1-n}\left(\frac{1}{\sqrt{1-n}} - 1\right) \qquad (D\text{-}10)$$

$B_{max\ n}$ is plotted in Fig. D-3. From the figure it follows that $A \leqslant 0.108$ for $n \leqslant 0.16$.

---

* $n > \cos^2\left(\frac{\theta}{2}\right) \qquad \therefore 1-n < 1 - \cos^2\left(\frac{\theta}{2}\right)$ or $< \sin^2\left(\frac{\theta}{2}\right)$

Thus, $\frac{1-n}{n} < \left[\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}\right]^2$ so that,

$$-\frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)} < \sqrt{\frac{1-n}{n}} < \frac{\sin\left(\frac{\theta}{2}\right)}{\cos\left(\frac{\theta}{2}\right)}$$

$$\therefore -2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) < \left[\sqrt{\frac{1-n}{n}}\; 2\cos^2\left(\frac{\theta}{2}\right)\right] < 2\sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right)$$

$$\therefore -\sin\theta < \left[\sqrt{\frac{1-n}{n}}\,(1+\cos\theta)\right] < \sin\theta$$

APPENDIX F

The cubic equation (4.17) and equation (4.55), (4.56) for the asymptotic error constants (a.e.c.s) are repeated below:

$$\lambda^3 + c\lambda - a = 0 \qquad \ldots \quad (F-1)$$

$$C_1 = \frac{1}{C_2} = \left| -\frac{2\lambda_{oe}^3}{a} \right| \quad ; \quad C_3 = \frac{1}{C_4} = \left| -\frac{3\lambda_{oe}^2}{c} \right| \qquad \ldots \quad (F-2)$$

where, $c = 1 + \frac{\cos\theta}{n}$ and $a = \frac{\sin\theta}{n}$

As the optimum root λ_{oe}, is real and has the same sign as a, Eqns. (F-2) may be written as

$$C_1 = \frac{1}{C_2} = \frac{2\lambda_{oe}^3}{a} \qquad C_3 = \frac{1}{C_4} = \frac{3\lambda_{oe}^2}{|c|} \qquad \ldots \quad (F-3)$$

Substituting λ_{oe} for λ in Eqn. (F-1) and rearranging,

$$c\,\lambda_{oe} = a\left(1 - \frac{\lambda_{oe}^3}{a}\right) \qquad \ldots \quad (F-4)$$

$$\therefore \quad \frac{c^3}{a^2}\,\frac{\lambda_{oe}^3}{a} = \left(1 - \frac{\lambda_{oe}^3}{a}\right)^3 \qquad \ldots \quad (F-5)$$

$$\therefore \quad \frac{1}{T} = \frac{c^3}{a^2} = \frac{\left(1 - \frac{C_1}{2}\right)^3}{\frac{C_1}{2}} = \frac{\left(1 - \frac{1}{2C_2}\right)^2}{\frac{1}{2C_2}} \qquad \ldots \quad (F-6)$$

Substituting λ_{oe} and rearranging Eqn. (F-1) once again,

$$\lambda_{oe}\left(\lambda_{oe}^2 + c\right) = a \qquad \ldots \quad (F-7)$$

$$\lambda_{oe}^2 \left(\lambda_{oe}^2 + c \right)^2 = a^2 \quad \ldots \quad (F\text{-}8)$$

$$\therefore \quad |c|^3 \frac{\lambda_{oe}^2}{|c|} \left(\frac{\lambda_{oe}^2}{|c|} + \frac{c}{|c|} \right)^2 = a^2 \ldots \quad (F\text{-}9)$$

$$\therefore \quad \frac{1}{T} = \frac{c^3}{a^2} = \frac{1}{\frac{|c|^3}{c^3} \frac{C_3}{3} \left(\frac{C_3}{3} + \frac{c}{|c|} \right)^2} \quad \ldots \quad (F\text{-}10)$$

$$\left.\begin{array}{ll} = \dfrac{1}{\frac{C_3}{3} \left(\frac{C_3}{3} + 1 \right)^2} & \text{for } c > 0 \\ \text{or } = \dfrac{-1}{\frac{C_3}{3} \left(\frac{C_3}{3} - 1 \right)^2} & \text{for } c < 0 \end{array}\right\} \quad (F\text{-}11)$$

Thus

$$\frac{1}{T} = \frac{c^3}{a^2} = \frac{1}{\pm \frac{C_3}{3} \left(\frac{C_3}{3} \pm 1 \right)^2} = \frac{1}{\pm \frac{1}{3C_4} \left(\frac{1}{3C_4} \pm 1 \right)^2} \quad (F\text{-}12)$$

APPENDIX G

Let

$$\lambda_o = \lambda_{oe} (1 + \mathcal{E}_\lambda) \qquad \ldots \quad \text{(G-1)}$$

where λ_{oe} is the exact value of the root. On substituting for λ_o, the denominator of F of Eqn. (5-2) becomes,

$$\text{Denom} = \left[n(1- \lambda_{oe}^2)-\cos\theta\right]^2 + \left[2n\lambda_{oe} - \sin\theta\right]^2$$

$$+ 2n\lambda_{oe}^2 \, \mathcal{E}_\lambda^2 (n+3n\lambda_{oe}^2 + \cos\theta) \qquad \ldots \quad \text{(G-2)}$$

$$= (\text{Denom})_e + 2n\lambda_{oe}^2 \, \mathcal{E}_\lambda^2 (n+3n\lambda_{oe}^2 + \cos\theta) \qquad \text{(G-3)}$$

$$= (\text{Denom})_e (1 + \mathcal{E}_D) \qquad \ldots \quad \text{(G-4)}$$

where the subscript e denotes exact value. It can be shown that $\mathcal{E}_D$ is below 0.03% for $n \geqslant 5$ as follows

$$\mathcal{E}_D = \frac{2n\lambda_{oe}^2 \, \mathcal{E}_\lambda^2 (n + 3n\lambda_{oe}^2 + \cos\theta)}{(\text{Denom})_e} \qquad \ldots \quad \text{(G-5)}$$

Consider the numerator. As $\lambda_{oe} \leqslant \dfrac{\sin\theta}{n + \cos\theta}$ from Eqn. (B-3) of Appendix B, the numerator is

$$\text{Num.} \leqslant 2n \left[\frac{\sin^2\theta}{(n+\cos\theta)^2}\right] \mathcal{E}_\lambda^2 \left[n+\cos\theta + \frac{3n\sin^2\theta}{(n+\cos\theta)^2}\right] \quad \text{(G-6)}$$

$$\text{or} \quad \leqslant 2n \left[\frac{\sin^2\theta}{(n+\cos\theta)}\right] \mathcal{E}_\lambda^2 \left[1 + \frac{3n\sin^2\theta}{(n+\cos\theta)^3}\right] \ldots \quad \text{(G-7)}$$

$$\text{or} \quad \leqslant 2n \left[\frac{\sin^2\theta}{n+\cos\theta}\right] \mathcal{E}_\lambda^2 \, (1 + 3T) \qquad \ldots \quad \text{(G-8)}$$

from Eqn. (4.26)

$$(\text{Num.})_{\max n} \leqslant 2\left(\frac{n \sin^2\theta}{n+\cos\theta}\right)_{\max n} \mathcal{E}_\lambda^2 \ (1 + 3T)_{\max n} \quad \ldots(G\text{-}9)$$

$$= \frac{4n\ (1-n^2+n\sqrt{n^2-1})}{\sqrt{n^2-1}}\ \mathcal{E}_\lambda^2 \ (1+3T_{\max n}) \quad \ldots(G\text{-}10)$$

$$= 4(n^2 - n\sqrt{n^2-1})\ \mathcal{E}_\lambda^2 \ (1 + 3T_{\max n}) \quad \ldots(G\text{-}11)$$

The factor $(n^2-n\sqrt{n^2-1})$ is positive and decreases monotonically. Its value is $\leqslant 0.505$ for $n \geqslant 5$. Also, $T_{\max n} \leqslant 0.044$ for $n \geqslant 5$ from Fig. C-1 in Appendix C. Thus,

$$(\text{Num.})_{\max} \leqslant 2.02 \times 1.132\ \mathcal{E}_\lambda^2 \quad \ldots(G\text{-}12)$$

$$\text{or} \leqslant 2.29\ \mathcal{E}_\lambda^2 \quad \ldots(G\text{-}13)$$

From Fig. 3.3, it is seen that for a given n, F is maximum, that is, Denom. is a minimum when $\theta = 0$. As the optimum root is zero at $\theta = 0$ (for $n \geqslant 1$) as seen from the cubic equation (4.1), then,

$$(\text{Denom})_{\min} = (\text{Denom.})_{\substack{\theta=0 \\ \lambda_0=0}} \quad \ldots(G\text{-}14)$$

$$= (n-1)^2 \quad \ldots(G\text{-}15)$$

Therefore,

$$(\text{Denom.})_{\min} \geqslant 16 \qquad \text{for } n \geqslant 5 \quad \ldots(G\text{-}16)$$

From Eqn. (G-5) and the inequalities (G-13) and (G-14),

$$(\mathcal{E}_D)_{\max} \leqslant \frac{(\text{Num.})_{\max}}{(\text{Denom.})_{\min}} \text{ or } \leqslant \frac{2.29}{16}\ \mathcal{E}_\lambda^2 \quad \ldots(G\text{-}17)$$

$$\text{or} \leqslant 0.028\ \% \quad \ldots(G\text{-}18)$$

as $\mathcal{E}_\lambda \leqslant 0.044$ for $n \geqslant 5$

APPENDIX H

The denominator of F of Eqn. (5.4) is,

$$(\text{Denom}) = \left[n-\cos\theta - \frac{n\sin^2\theta}{(n+\cos\theta)^2}\right]^2 + \left[\frac{2n\sin\theta}{n+\cos\theta} - \sin\theta\right]^2 \qquad \ldots \quad \text{(H-1)}$$

$$= (n-\cos\theta)^2 - \frac{2n\sin^2\theta(n-\cos\theta)}{(n+\cos\theta)^2} + \frac{n^2\sin^4\theta}{(n+\cos\theta)^4} + \frac{\sin^2\theta(n-\cos\theta)^2}{(n+\cos\theta)^2} \qquad \ldots \quad \text{(H-2)}$$

$$= \left(\frac{n-\cos\theta}{n+\cos\theta}\right)\left[(n-\cos\theta)(n+\cos\theta) - \frac{2n\sin^2\theta}{n+\cos\theta} + \frac{\sin^2\theta\,(n-\cos\theta)}{n+\cos\theta}\right] + \frac{n^2\sin^4\theta}{(n+\cos\theta)^4} \qquad \ldots \quad \text{(H-3)}$$

$$= \left(\frac{n-\cos\theta}{n+\cos\theta}\right)\left(n^2-\cos^2\theta - \sin^2\theta\right) + \frac{n^2\sin^4\theta}{(n+\cos\theta)^4} \qquad \ldots \quad \text{(H-4)}$$

$$\simeq \left(\frac{n-\cos\theta}{n+\cos\theta}\right)(n^2-1) \qquad \ldots \quad \text{(H-5)}$$

as $\mathcal{E}_D = \frac{n^2\sin^4\theta}{(n+\cos\theta)^4}\left(\frac{n+\cos\theta}{n-\cos\theta}\right)\frac{1}{(n^2-1)}$ is << 1 for n > 5.
This is seen as follows:

$$\mathcal{E}_{D_{\max n}} = \left[\frac{n^2\sin^4\theta}{(n+\cos\theta)^2(n^2-\cos^2\theta)(n^2-1)}\right]_{\max n} \qquad \text{(H-6)}$$

$$< \frac{\left[n^2\sin^4\theta\right]_{\max n}}{\left[(n+\cos\theta)^2(n^2-\cos^2\theta)(n^2-1)\right]_{\min n}} \qquad \text{(H-7)}$$

or $$< \frac{n^2}{(n-1)^2(n^2-1)^2} \qquad \ldots \quad \text{(H-8)}$$

The expression (H-8) decreases monotonically from ∞ (at n = 1) to 0 (at $n = \infty$) and is therefore, <0.0027 for n $\geqslant$ 5.

Thus $\mathcal{E}_D < 0.27\,\%$ for $n \geqslant 5$. As this error is small, the error $\mathcal{E}_F$ in F given by

$$F = \frac{n}{\text{Denom.}} \simeq \frac{n}{n^2-1}\left(\frac{n+\cos\theta}{n-\cos\theta}\right) \qquad \ldots(\text{H-9})$$

is $\approx \mathcal{E}_D$.

APPENDIX I

Let

$$n = (1+\epsilon)\Pi = (1+\epsilon)\cos^2\left(\frac{\theta}{2}\right) \qquad \ldots(\text{I-1})$$

Substituting for n in Eqn. (4.46) and retaining only 1st order terms,

$$\lambda_o = \sqrt{\frac{1-\Pi-\Pi\epsilon}{\Pi(1+\epsilon)}}\left\{1+\frac{1}{2(1-n)}\left[\frac{\sin\theta}{\sqrt{\frac{1-\Pi-\Pi\epsilon}{\Pi(1+\epsilon)}}}-(1+\cos\theta)\right]\right\} \qquad \ldots(\text{I-2})$$

$$\simeq \tan\left(\frac{\theta}{2}\right)\left[1-\frac{\epsilon}{2}-\frac{\Pi}{1-\Pi}\frac{\epsilon}{2}\right] \times$$

$$\left\{1+\frac{2\cos^2\left(\frac{\theta}{2}\right)}{2(1-\Pi)}\left[1+\frac{\epsilon}{2}+\frac{\Pi}{1-\Pi}\frac{\epsilon}{2}-1\right]\right\} \qquad (\text{I-3})$$

$$\simeq \tan\left(\frac{\theta}{2}\right)\left[1-\frac{\epsilon}{2}+\frac{\Pi^2}{(1-\Pi)^2}\frac{\epsilon}{2}\right] \qquad \ldots(\text{I-4})$$

$$\simeq \tan\left(\frac{\theta}{2}\right)\left(1-\frac{\epsilon}{2}\right)\left[1+\frac{\Pi^2}{(1-\Pi)^2}\frac{\epsilon}{2}\right] \qquad \ldots(\text{I-5})$$

$$\simeq \lambda_{oe}\left[1+\frac{\Pi^2}{(1-\Pi)^2}\frac{\epsilon}{2}\right] \qquad \ldots(\text{I-6})$$

from Eqn. (J-10) of Appendix J

Therefore,

$$\varepsilon_\lambda = \frac{\lambda_o - \lambda_{oe}}{\lambda_{oe}} = \frac{\Pi^2}{(1-\Pi)^2}\frac{\epsilon}{2}$$

and

$$\frac{1-\Pi}{\Pi} \, 4 \frac{\mathcal{E}_\lambda^2}{\epsilon^2} = \frac{\Pi^3}{(1-\Pi)^3}$$

which $\rightarrow \dfrac{n^3}{(1-n)^3}$ as $\epsilon \rightarrow 0$. Hence,

$$\underset{\epsilon \rightarrow 0}{\mathrm{Lt}} \frac{1-\Pi}{\Pi} \frac{\mathcal{E}_\lambda^2}{\epsilon^2} = \frac{n^3}{(1-n)^3} < 0.00137 \text{ or } 0.137\ \%$$

for $n \leqslant 0.1$

APPENDIX J

The Taylor's series expansion for λ_o around the value $\tan\left(\frac{\theta}{2}\right)$ corresponding to $n = \Pi = \cos^2\left(\frac{\theta}{2}\right)$ is given by

$$\lambda_o\Big|_{n=\Pi(1+\epsilon)} = \lambda_o\Big|_{n=\Pi} + \frac{d\lambda_o}{dn}\Big|_{n=\Pi} \times \Pi\,\epsilon + \frac{d^2\lambda_o}{dn^2}\Big|_{n=\Pi} \Pi^2\epsilon^2 + \ldots \qquad \ldots(J\text{-}1)$$

From Eqn. (5.8),

$$\lambda_o\Big|_{n=\Pi} = \tan\left(\frac{\theta}{2}\right) \qquad \ldots(J\text{-}2)$$

Now consider the cubic equation (4.1) which is slightly modified here;

$$n\ \lambda^3 + (n + \cos\theta)\lambda - \sin\theta = 0 \qquad \ldots(J\text{-}3)$$

Three successive differentiations with respect to n yield respectively,

$$(3n\lambda^2 + n + \cos\theta)\ \lambda' + \lambda(1 + \lambda^2) = 0 \qquad \ldots(J\text{-}4)$$

$$(3n\lambda^2 + n + \cos\theta)\ \lambda'' + 2(1+3\lambda^2 + 3n\lambda\lambda')\ \lambda' = 0 \qquad (J\text{-}5)$$

and

$$(3n\lambda^2 + n + \cos\theta)\ \lambda''' + 3(1+3\lambda^2 + 6n\lambda\lambda')\ \lambda'' + 6(3\lambda + n\lambda')\ \lambda'^2 = 0 \qquad \ldots(J\text{-}6)$$

From Eqns (J-2) and (J-4),

$$\frac{d\lambda_o}{dn}\Big|_{n=\Pi} = -\tan\left(\frac{\theta}{2}\right)\frac{\epsilon}{2} \qquad \ldots(J\text{-}7)$$

Similarly,

$$\left.\frac{d^2 \lambda_o}{dn^2}\right|_{n = \Pi} \times \frac{\Pi^2 \epsilon^2}{2} = \tan\left(\frac{\theta}{2}\right) (3-\Pi) \frac{\epsilon^2}{8} \qquad \ldots(J\text{-}8)$$

$$\left.\frac{d^3 \lambda_o}{dn^3}\right|_{n = \Pi} \times \frac{\Pi^3 \epsilon^3}{6} = \tan\left(\frac{\theta}{2}\right) (\Pi^2+2\Pi-5) \frac{\epsilon^3}{16} \qquad \ldots(J\text{-}9)$$

Hence,

$$\left.\lambda_o\right|_{n=\Pi(1+\epsilon)} = \tan\left(\frac{\theta}{2}\right) \left[1 - \frac{1}{2}\epsilon + \frac{3-\Pi}{8}\epsilon^2 + \frac{\Pi^2+2\Pi-5}{16}\epsilon^3 + \ldots\right] \qquad \ldots(J\text{-}10)$$

If an approximation is made by ignoring ϵ^3 and higher order terms,

$$\lambda_o \approx \tan\left(\frac{\theta}{2}\right) \left[1 - \frac{1}{2}\epsilon + \frac{3-\Pi}{8}\epsilon^2\right] \qquad \ldots(J\text{-}11)$$

Denoting the exact value as given by Eqn. (J-10) by λ_{oe}, the error in λ_o is given by

$$\mathcal{E}_\lambda = \frac{\lambda_o - \lambda_{oe}}{\lambda_{oe}} \approx \frac{(\Pi^2+2\Pi-5)\epsilon^3}{16\left(1-\frac{\epsilon}{2}\right)} \qquad \ldots(J\text{-}12)$$

As $0 \leqslant \Pi \leqslant 1$, $\mathcal{E}_\lambda$ is negative. The right hand side of Eqn. (J-12) has a maximum magnitude at $\mathbf{o} = \Pi$. Thus,

$$|\mathcal{E}_\lambda| \leqslant \frac{5}{16} \frac{\epsilon^3}{\left(1-\frac{\epsilon}{2}\right)} \leqslant 0.01 \text{ or } 1\,\% \qquad \ldots(J\text{-}13)$$

for $\epsilon \leqslant 0.3$

APPENDIX K

When $n = \prod (1+\epsilon)$ or $\cos^2\left(\frac{\theta}{2}\right)(1+\epsilon)$, λ_o is given by Eqn. (5.8). Substituting for n and λ_o and using ϕ in place of $\frac{\theta}{2}$, F of Eqn. (5.2) becomes,

$$F = \frac{\cos^2\phi\ (1+\epsilon)}{D_1^2 + D_2^2} \qquad \ldots(K-1)$$

where,

$$D_1 = (1+\epsilon)\cos^2\phi - (1+\epsilon)\sin^2\phi\left[1-\epsilon + \frac{4-\cos^2\phi}{4}\epsilon^2 + \frac{\cos^4\phi + 3\cos^2\phi - 8}{8}\epsilon^3 + \ldots\right] - 2\cos^2\phi + 1 \qquad \ldots(K-2)$$

$$= \left[\cos^2\phi\right]\epsilon + \left[\cos^2\phi\sin^2\phi\right]\frac{\epsilon^2}{4} - \left[\sin^2\phi\cos^2\phi(1+\cos^2\phi)\right]\frac{\epsilon^3}{8} + \ldots \qquad \ldots(K-3)$$

$$= \left[\cos^2\phi\right]\epsilon\left[1 + \frac{\sin^2\phi}{4}\epsilon - \sin^2\phi\left(\frac{1+\cos^2\phi}{8}\right)\epsilon^2 + \ldots\right] \qquad \ldots(K-4)$$

and,

$$D_2 = 2(1+\epsilon)\cos^2\phi\tan\phi\left[1-\epsilon + \frac{3-\cos^2\phi}{8}\epsilon^2 + \frac{\cos^4\phi + 2\cos^2\phi - 5}{16}\epsilon^3 + \ldots\right] - 2\sin\phi\cos\phi \qquad \ldots(K-5)$$

$$= \left[\sin\phi\cos\phi\right]\epsilon\left[1 - \frac{1+\cos^2\phi}{4}\epsilon + \frac{1+\cos^4\phi}{8}\epsilon^2 + \ldots\right] \qquad \ldots(K-6)$$

Thus,

$$D_1^2 + D_2^2 = \left[\cos^2 \phi\right] \epsilon^2 \left[1 - \frac{1+\cos^2\phi}{2}\epsilon + \frac{(1-\cos^2 \phi)(5-\cos^2 \phi)}{16}\epsilon^2 + ..\right] \quad \text{(K-7)}$$

Therefore,

$$F = \frac{1}{\epsilon^2}(1+\epsilon)\left[1+\frac{1-\cos^2 \phi}{2}\epsilon - \frac{(1-\cos^2 \phi)(1+3\cos^2 \phi)}{16}\epsilon^3 + \ldots\right] \text{(K-8)}$$

$$= \frac{1}{\epsilon^2}\left[1+\frac{3-\cos^2\phi}{2}\epsilon + \frac{(1-\cos^2 \phi)(7-3\cos^2 \phi)}{16}\epsilon^2 + \ldots\right] \text{(K-9)}$$

If ϵ^2 and the higher order terms are ignored,

$$F \simeq \frac{1}{\epsilon^2}\left[1 + \frac{3-\cos^2 \phi}{2}\epsilon\right] \quad \ldots\text{(K-10)}$$

Denoting the exact value as given by Eqn. (K-9) by F_e the error in F is given by

$$\mathcal{E}_F = \frac{F-F_e}{F_e} \simeq \frac{(1-\cos^2 \phi)(7-3\cos^2 \phi)\,\epsilon^2}{16\left[1 + \frac{3-\cos^2 \phi}{2}\epsilon\right]} \quad \ldots\text{(K-11)}$$

$$\leqslant \frac{\left[(1-\cos^2 \phi)(7-3\cos^2 \phi)\,\epsilon^2\right]_{max}}{16\left[1+ \frac{3-\cos^2 \phi}{2}\epsilon\right]_{min}} \quad \ldots\text{(K-12)}$$

$$\text{or} \quad \leqslant \frac{7\epsilon^2}{16(1+\epsilon)} \quad \ldots\text{(K-13)}$$

Hence,

$$\mathcal{E}_F \leqslant 0.03 \text{ or } 3\,\% \quad \ldots\text{(K-14)}$$

for $\epsilon \leqslant 0.3$ or $n \leqslant 1.3\,\Pi$

APPENDIX L

Let λ_o be the calculated value and λ_{oe} the exact value of the required root. Then the error in λ_o is given by

$$\mathcal{E}_\lambda = \frac{\lambda_o - \lambda_{oe}}{\lambda_{oe}} \qquad \text{...(L-1)}$$

Let σ_S be the calculated value of the imaginary part of source termination, and σ_{Se} the exact value. Then error in σ_S is given by,

$$\mathcal{E}_{\sigma_S} = \frac{\sigma_S - \sigma_{Se}}{\sigma_{Se}} = \frac{(\lambda_o - \lambda_{oe})\rho_1}{\lambda_{oe}\rho_1 - \sigma_{11}} \qquad \text{...(L-2)}$$

from Eqns. (3.27) and (3.21). $\mathcal{E}_{\sigma_S}$ can be rearranged in terms of $\mathcal{E}_\lambda$ as,

$$\mathcal{E}_{\sigma_S} = \frac{\mathcal{E}_\lambda}{1 - \frac{\sigma_{11}}{\rho_1}\frac{1}{\lambda_{oe}}} = \frac{\mathcal{E}_\lambda}{1 - \frac{\sigma_{11}}{\rho_{11}}\sqrt{\frac{n_i}{n}}\frac{1}{\lambda_{oe}}} \qquad \text{...(L-3)}$$

As $\lambda_{oe} \simeq \lambda_o$ (within 1.5%) therefore,

$$\mathcal{E}_{\sigma_S} \simeq \frac{\mathcal{E}_\lambda}{1 - \sqrt{\frac{n_i}{n}}\frac{1}{\rho_{11}}\frac{\sigma_{11}}{\lambda_o}} \qquad \text{...(L-4)}$$

Similarly,

$$\mathcal{E}_{\sigma_L} \simeq \frac{\mathcal{E}_\lambda}{1 - \sqrt{\frac{n_i}{n}}\frac{1}{\rho_{22}}\frac{\sigma_{22}}{\lambda_o}} \qquad \text{...(L-5)}$$

Here, $|\mathcal{E}_\lambda| < 1.5\%$. Three situations can be considered.

Case 1: where $\frac{\sigma_{11}}{\lambda_o} \leqslant 0$ and $\frac{\sigma_{22}}{\lambda_o} \leqslant 0$

Here $|\mathcal{E}_\sigma| \leqslant |\mathcal{E}_\lambda|$ so that $|\mathcal{E}_\sigma| < 1.5\%$

Case 2: where $\sqrt{\frac{n_i}{n}}\,\frac{1}{\rho_{11}}\,\frac{\sigma_{11}}{\lambda_o} \geqslant 2$ and $\sqrt{\frac{n_i}{n}}\,\frac{1}{\rho_{22}}\,\frac{\sigma_{22}}{\lambda_o} \geqslant 2$.

Here again $|\mathcal{E}_\sigma| \leqslant |\mathcal{E}_\lambda|$ so that $|\mathcal{E}_\sigma| < 1.5\%$.

Case 3: where $0 < \left(\sqrt{\frac{n_i}{n}}\,\frac{1}{\rho_{11}}\,\frac{\sigma_{11}}{\lambda_o}\right) < 2$ and/or $0 < \left(\sqrt{\frac{n_i}{n}}\,\frac{1}{\rho_{22}}\,\frac{\sigma_{22}}{\lambda_o}\right) < 2$

Here one or both $|\mathcal{E}_\sigma| > |\mathcal{E}_\lambda|$. Thus one or both $|\mathcal{E}_\sigma| > 1.5\%$ and may approach ∞ as the quantities in brackets approach unity.

APPENDIX M

Let $k = 1 + \epsilon$ where $\epsilon << 1$. Now, the cubic equation for $k = 1$ becomes

$$\lambda^3 + \left[1 + \frac{2\cos\theta}{1+\cos\theta}\right]\lambda - \frac{2\sin\theta}{1+\cos\theta} = 0 \qquad \text{(M-1)}$$

and the root is $\lambda_o = \tan\left(\frac{\theta}{2}\right)$. Then

$$\lambda_o\Big|_{k=1+\epsilon} \simeq \lambda_o\Big|_{k=1} + \epsilon \cdot \frac{d\lambda_o}{dk}\Big|_{k=1} \qquad \text{(M-2)}$$

By differentiating Eqn. (M-1) it can be shown that

$$\frac{d\lambda_o}{dk}\Big|_{k=1} = -\frac{1}{2}\tan\left(\frac{\theta}{2}\right) \qquad \text{(M-3)}$$

$$\therefore \quad \lambda_o\Big|_{k=1+\epsilon} = \tan\left(\frac{\theta}{2}\right)\left[1 - \frac{\epsilon}{2}\right] \qquad \text{(M-4)}$$

$$\simeq \frac{\tan\left(\frac{\theta}{2}\right)}{\sqrt{k}} \qquad \text{(M-5)}$$

Similarly, by considering the cubic equation involving it can be shown that

$$\lambda_o\Big|_{\eta=1+\epsilon} \simeq \tan\left(\frac{\theta}{2}\right)\left[1 - \frac{\epsilon}{2(1+\cos\theta)}\right] \qquad \text{(M-6)}$$

$$\simeq \frac{\tan\left(\frac{\theta}{2}\right)}{\sqrt{1+\frac{\epsilon}{1+\cos\theta}}} \qquad \text{(M-7)}$$

$$= \frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{\eta + \cos\theta}} \qquad \text{(M-8)}$$

provided that $\dfrac{\epsilon}{2(1+\cos\theta)} << 1$. This condition is not satisfied for θ very close to 180°. But expression (M-8) still holds as the value of λ_o at $\theta = 180^{\circ}$ is given by

$$\lambda_o\Big|_{\theta=180^{\circ}} = \sqrt{-1+\frac{2}{\eta-1}} \simeq \sqrt{\frac{2}{\eta-1}} = \frac{\sqrt{2}\sin\left(\frac{\theta}{2}\right)}{\sqrt{\eta+\cos\theta}}\Bigg|_{\theta=180^{\circ}} \qquad \text{(M-9)}$$

Thus expression (M-8) is a valid approximation for η close to unity. Similarly it can be shown that for $S \simeq 1$ and $\delta \simeq 0$

$$\lambda_o \simeq \frac{2\sqrt{S}\sin\left(\frac{\theta}{2}\right)}{\sqrt{4\Pi S+(S-1)^2}} \qquad \text{(M-10)}$$

and

$$\lambda_o \simeq \frac{\sin\left(\frac{\theta}{2}\right)}{\sqrt{\delta+\Pi}} \qquad \text{(M-11)}$$

Date Due

EE-1970-M-NAM-DES